DIFFERENTIAL ALGEBRA SPACE TOOLBOX* FOR NONLINEAR UNCERTAINTY PROPAGATION IN SPACE DYNAMICS

6th International Conference on Astrodynamics Tools and Techniques (ICATT)
Darmstadt, Germany

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P. Di Lizia, R. Armellin, C. Y. Valles and
G. Ortega

March 17th, 2016

** Developed under ITT AO/1-7570/13/NL/MH: Nonlinear Propagation of Uncertainties in Space Dynamics based on Taylor Differential Algebra"*

MOTIVATION

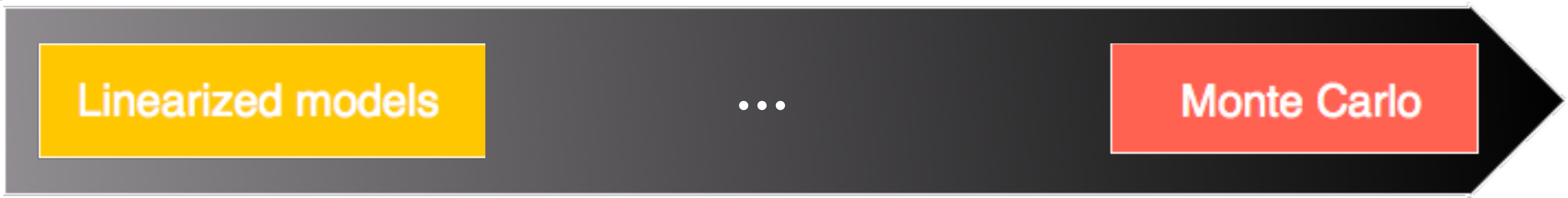
- Uncertainty propagation is a crucial issue in spaceflight dynamics
  - Space surveillance and tracking
  - Reentry and casualty area computation
  - Robust design of space trajectories and systems
  - …

- Most spaceflight mechanics problems involve nonlinear behavior

↓

Need of efficient tools for nonlinear propagation of uncertainties

PROPOSED APPROACH



- low computational burden
- low accuracy

- computationally intensive
- high accuracy

Can we find a compromise technique?

- We need a technique to:
  - Improve accuracy of linearized models
  - Reduce computational cost of classical Monte Carlo

Proposed Approach

Linearized models … Differential Algebra … Monte Carlo

- low computational burden
- low accuracy

- computationally intensive
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Can we find a compromise technique?

- We need a technique to:
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Differential Algebra

AST

DIFFERENTIAL ALGEBRA

- Differential Algebra (DA) is an automatic differentiation technique

Algebra of real numbers → Algebra of Taylor polynomials

- DA can be implemented in a computer environment (DACE)
- Given any sufficiently regular function $f(x)$
  - Initialize x as a DA variable: $[x] = \overline{x} + \delta x$
  - Evaluate f in the DA framework:

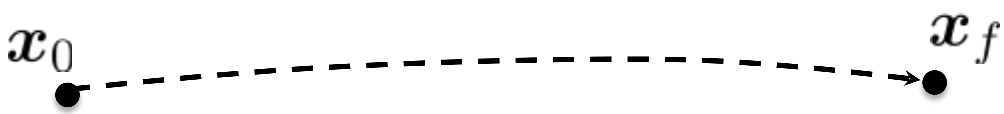
$$f([x]) = \mathcal{T}_f(\delta x)$$

→ Taylor expansion of f around $\overline{x}$ up to an **arbitrary order** k

Expansion of the Flow of ODEs

- Given any dynamics

$$\dot{x} = f(x, t)$$



- Any numerical integrator is based on the evaluation of f and its algebraic manipulation
  - Initialize x_0 as DA vector
  - Perform operations in DA

⇨ Taylor expansion of x_f w.r.t. x_0

- Uncertainty propagation can benefit from DA in different ways

EXPANSION OF THE FLOW OF ODES

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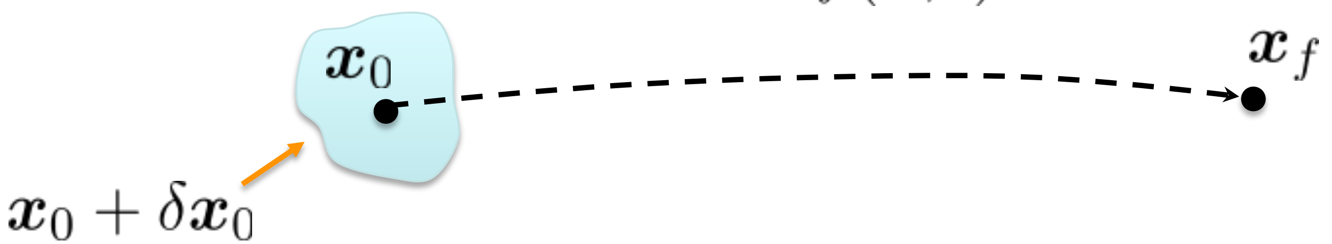

- Any numerical integrator is based on the evaluation of f and its algebraic manipulation
  - Initialize $\boldsymbol{x}_0$ as DA vector
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⇒ Taylor expansion of $\boldsymbol{x}_f$ w.r.t. $\boldsymbol{x}_0$

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Expansion of the Flow of ODEs

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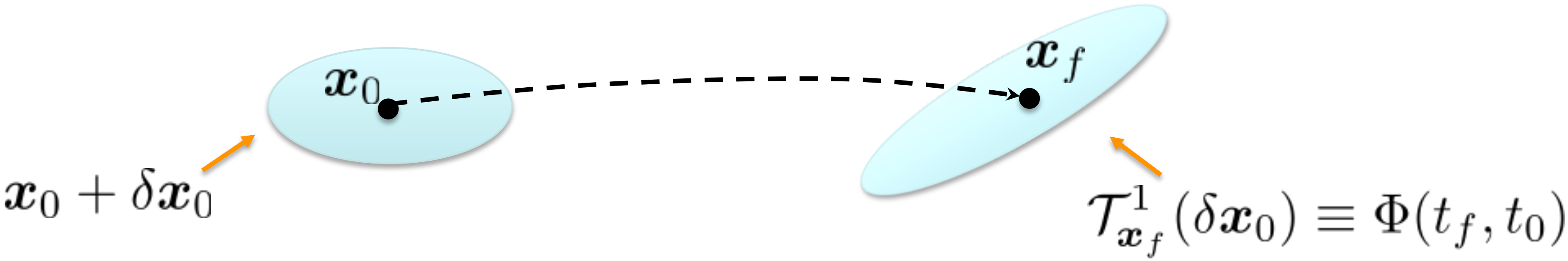
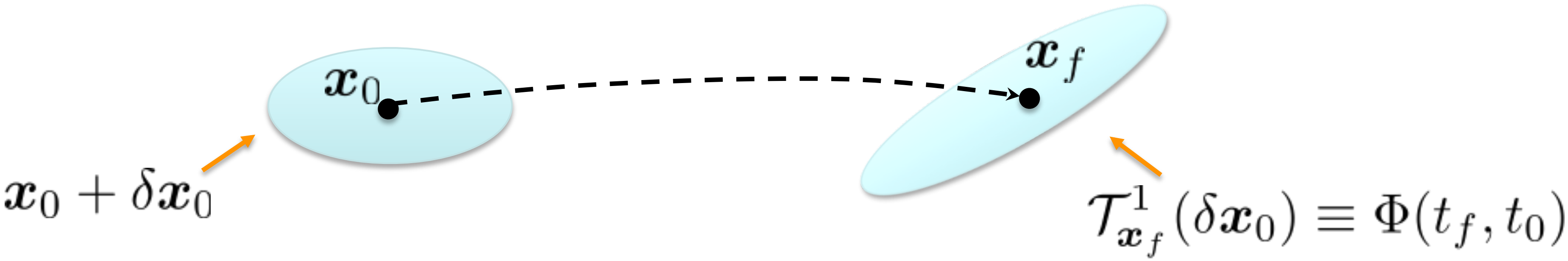
Linear Covariance Propagation



- If all computations are performed to order 1, the final Taylor expansion $\mathcal{T}^1_{x_f}(\delta x_0)$ coincides with the state-transition matrix $\Phi(t_f, t_0)$

Perform a single DA integration with expansion order 1 → Extract the STM from the Taylor polynomial map → Map the initial covariance: $C_f = \Phi\, C_0\, \Phi^T$

LINEAR COVARIANCE PROPAGATION



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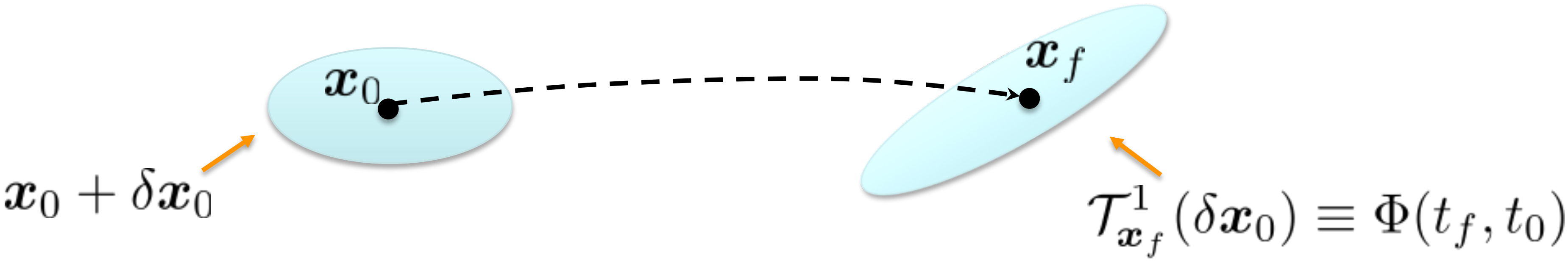
Linear Covariance Propagation



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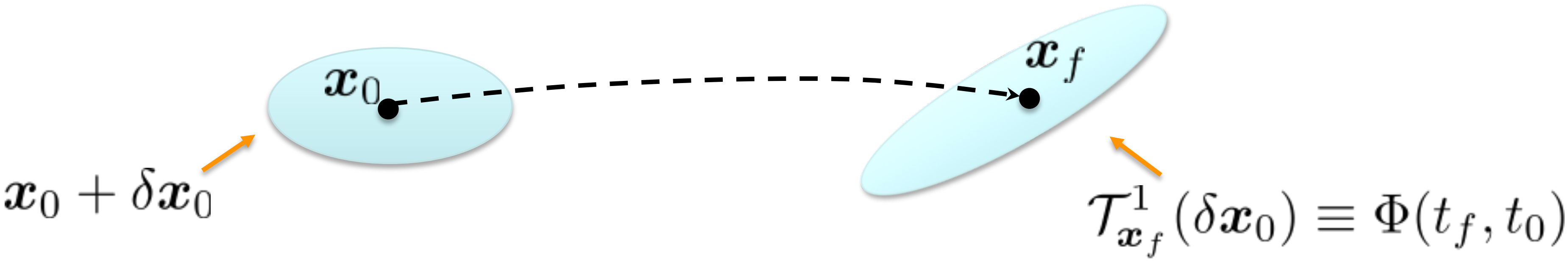

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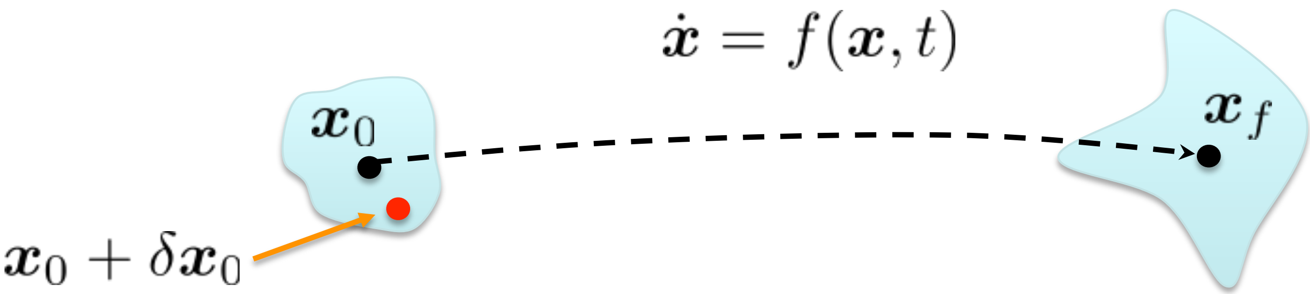
Perform a single DA integration with expansion order 1 → **Extract the STM from the Taylor polynomial map** → **Map the initial covariance:** $C_f = \Phi \, C_0 \, \Phi^T$

No need of variational equations!

DA-BASED MONTE CARLO



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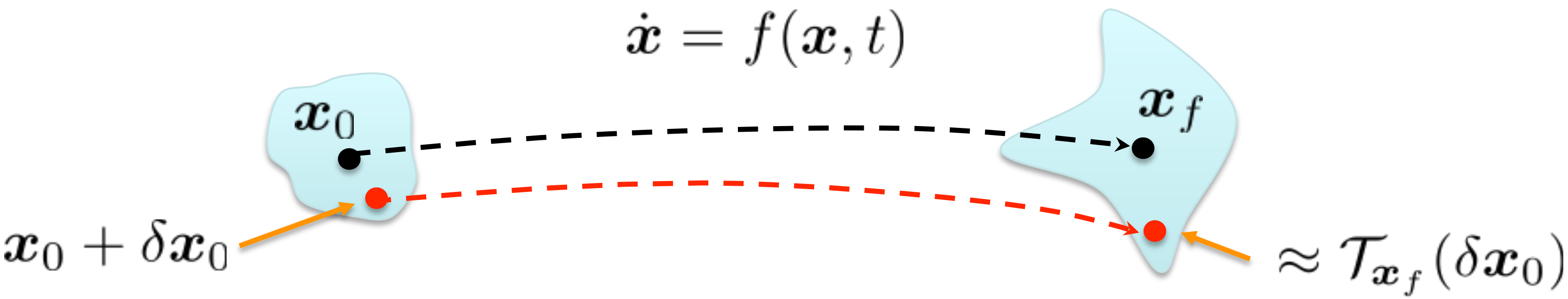


- Any pointwise integration can be replaced by the evaluation of the polynomial $\mathcal{T}_{x_f}(\delta x_0)$

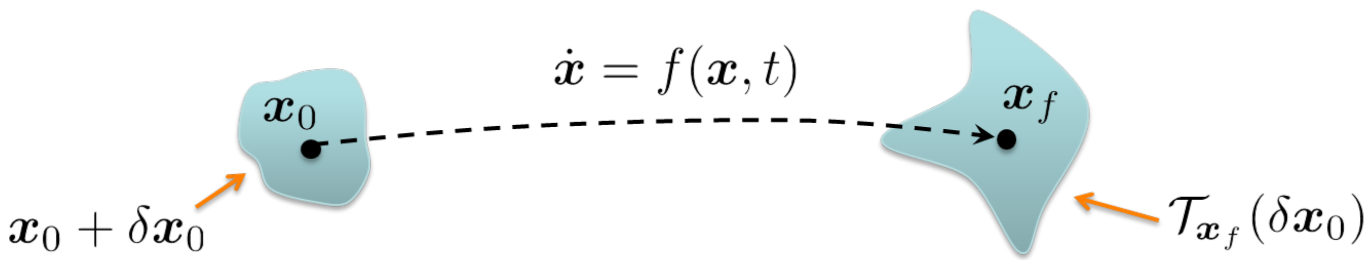
⇩

Saving in computational time w.r.t. classical MC

DA-BASED MONTE CARLO

Compute high order expansion of the flow → Evaluate expansion at samples → Compute moments of PDF

Generate random samples ↓ Evaluate expansion at samples



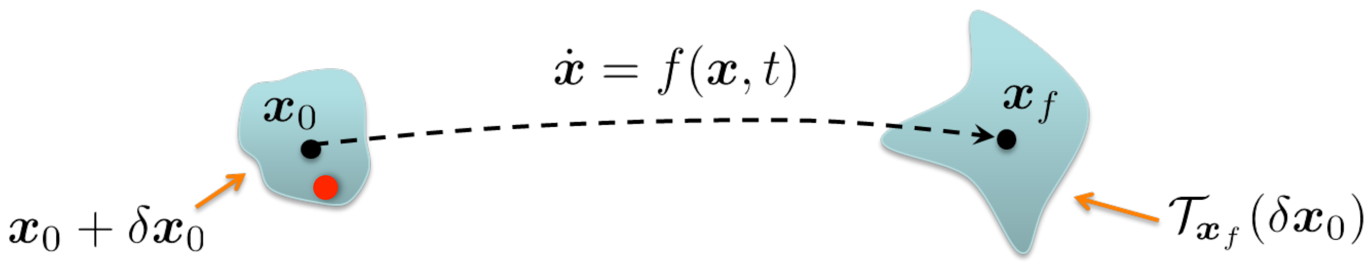
DA-based Monte Carlo

Generate random samples

Compute high order expansion of the flow

Evaluate expansion at samples

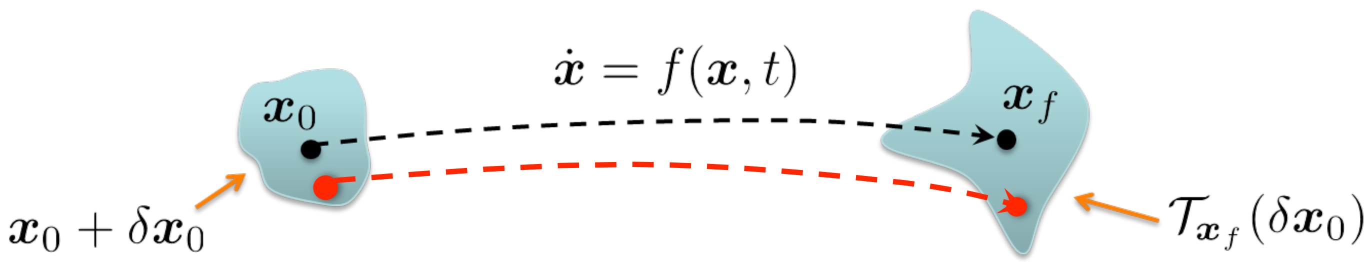
Compute moments of PDF



DA-based Monte Carlo

Generate random samples

Compute high order expansion of the flow → **Evaluate expansion at samples** → Compute moments of PDF



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Generate random samples

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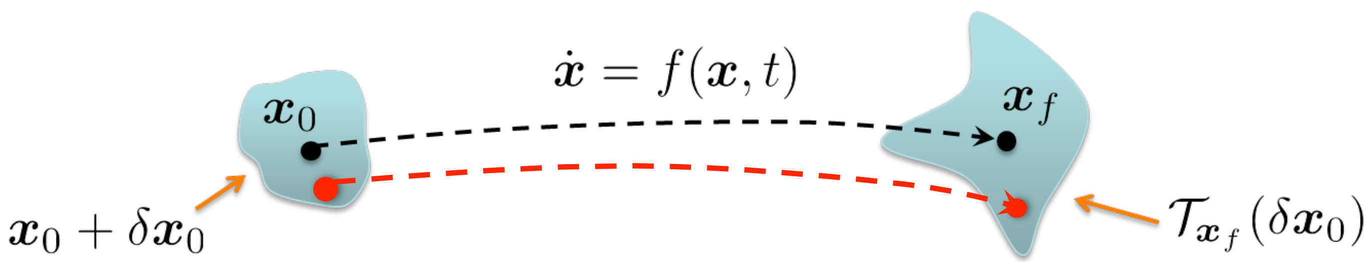

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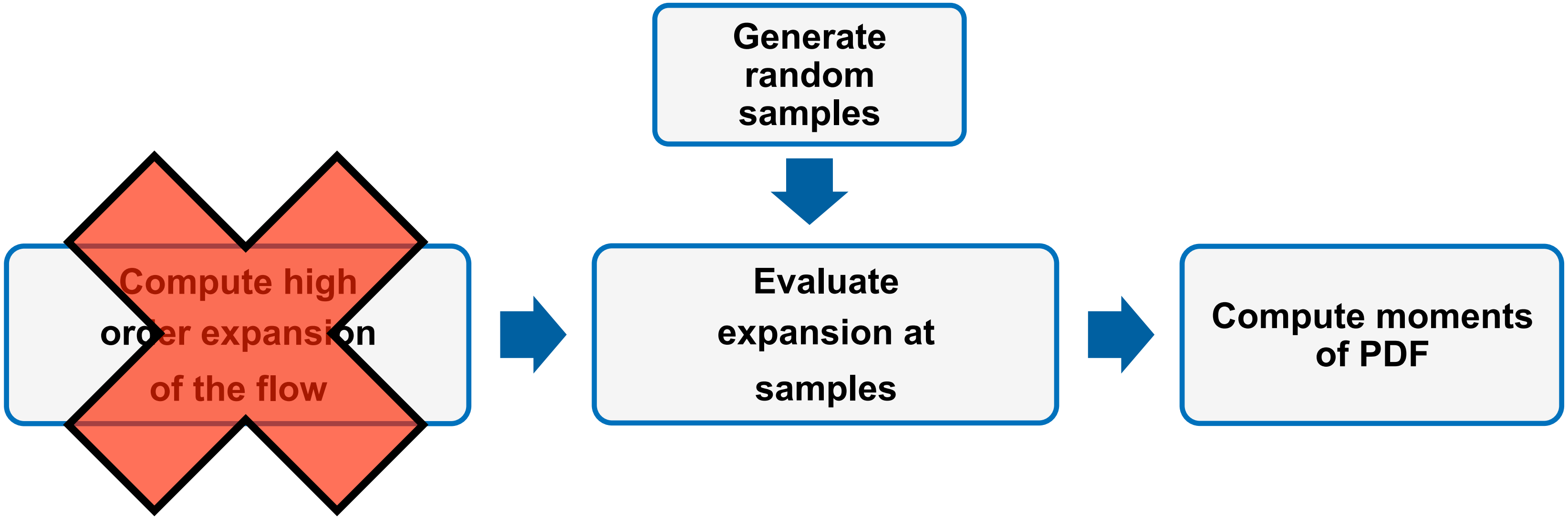
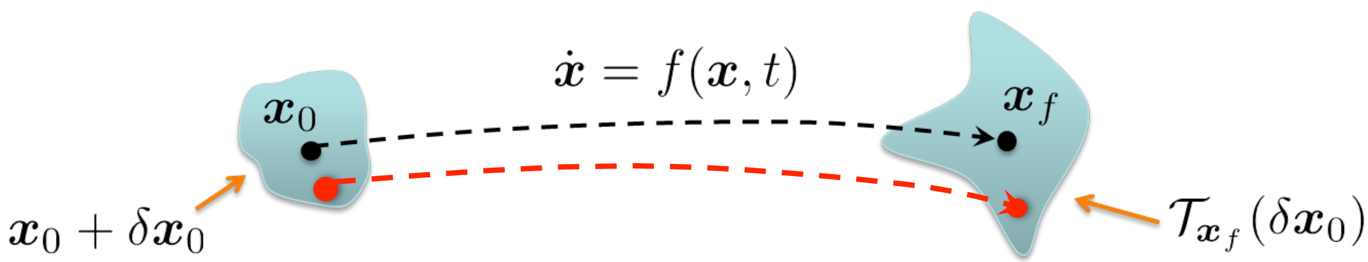
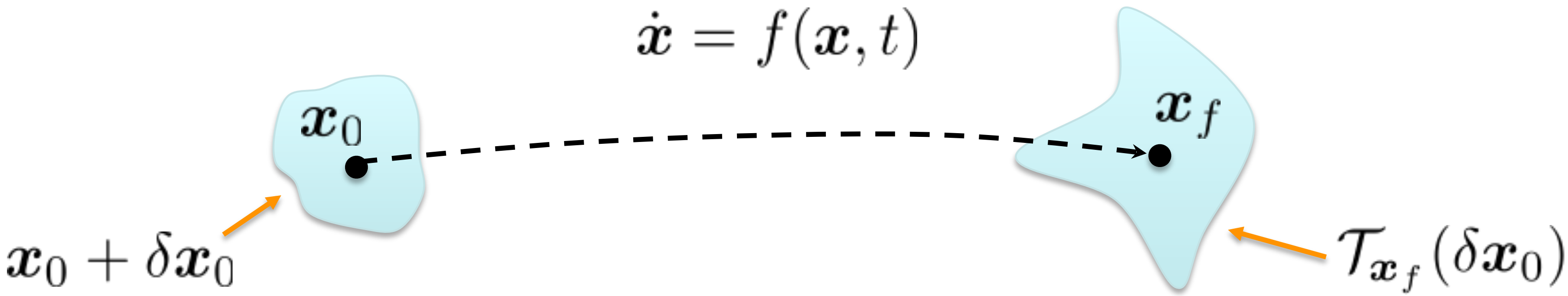




Same polynomial can be used to map different uncertainties

POLYNOMIAL BOUNDER



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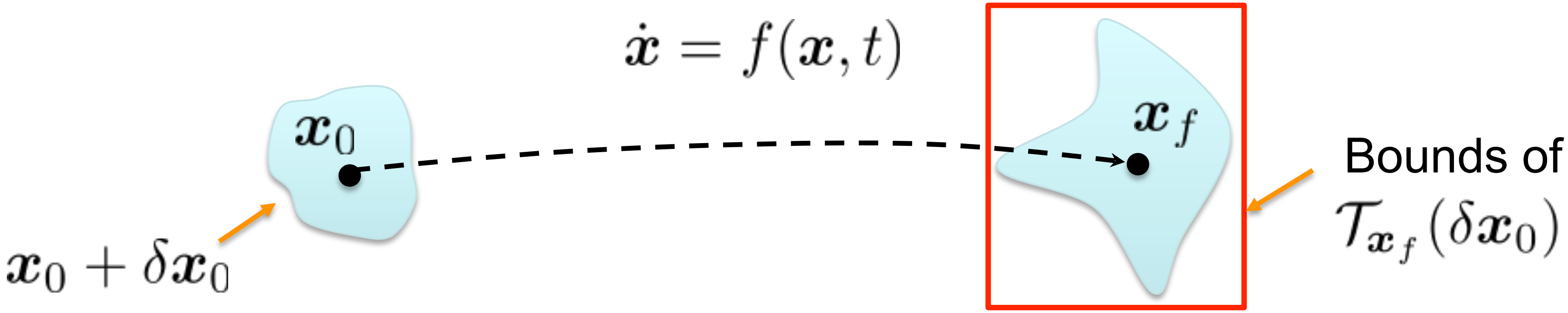
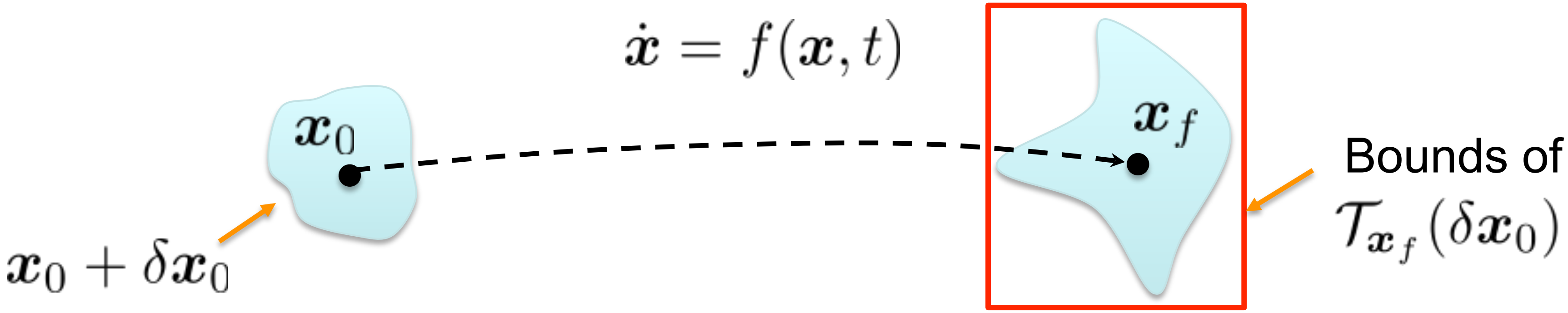


- Use polynomial bounders to estimate the range of the propagated uncertainties by bounding $\mathcal{T}_{\boldsymbol{x}_f}(\delta\boldsymbol{x}_0)$

POLYNOMIAL BOUNDER



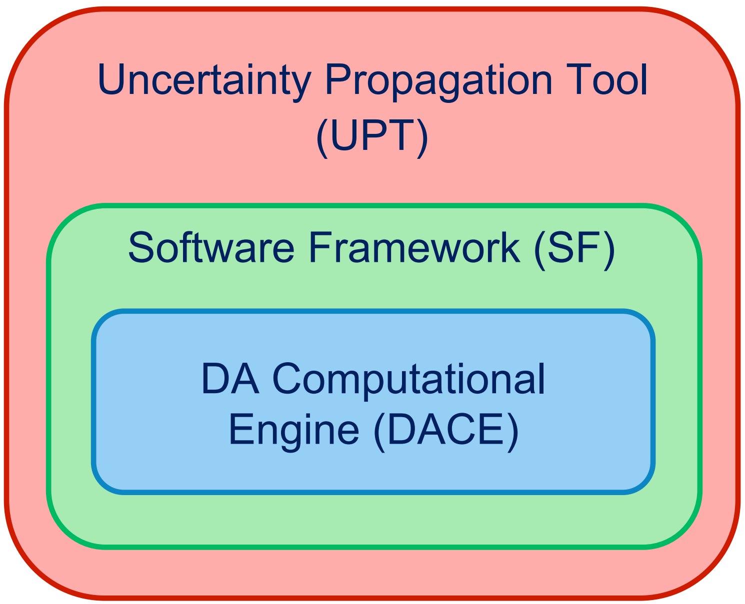
- Use polynomial bounders to estimate the range of the propagated uncertainties by bounding $\mathcal{T}_{x_f}(\delta x_0)$

Rescale the domain of each variable to [-1,1] → **Perform a single DA integration** → **Compute the range of the polynomial**

- Useful in some applications to verify constraints satisfaction (no need to map a statistical distribution)

DIFFERENTIAL ALGEBRA SPACE TOOLBOX

Differential Algebra
Space Toolbox



- **DA Computational Engine**

  Implements Taylor DA arithmetic to handle polynomial operations

- **Software Framework**

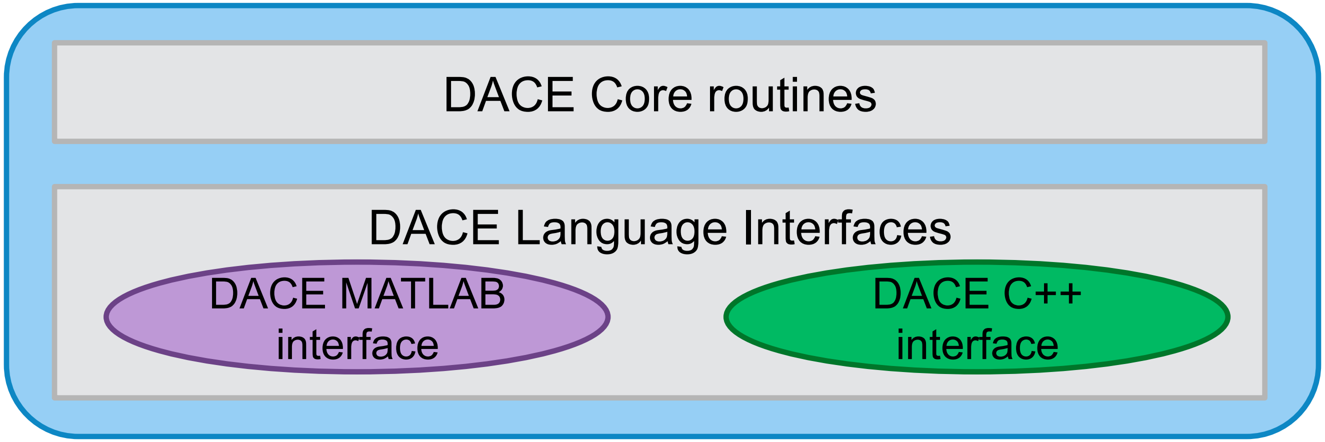
  Provides all routines to perform DA based propagation in astrodynamics

- **Uncertainty Propagation Tool**

  Provides all routines and an interface for DA-based propagation of uncertainties

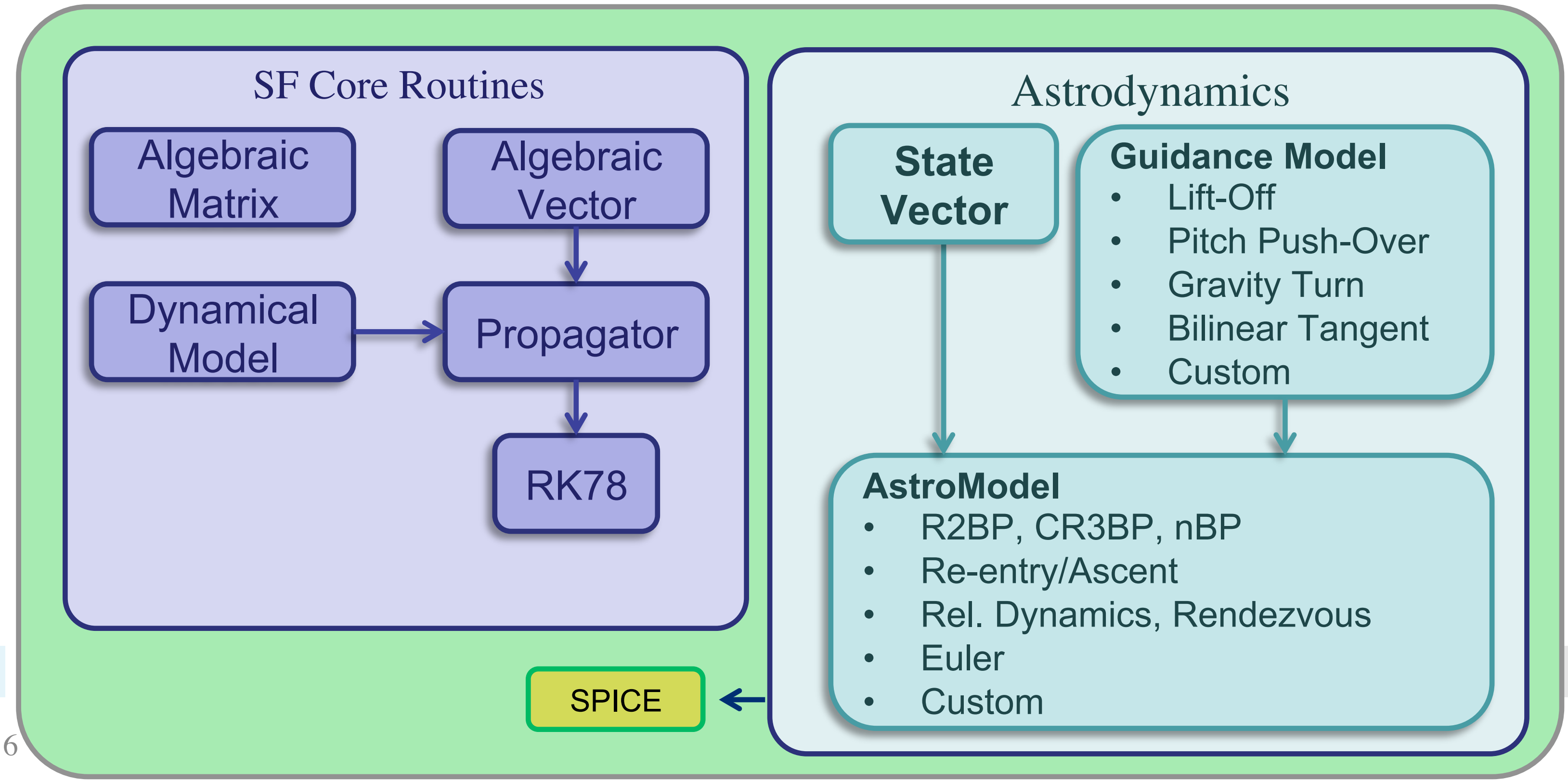
DACE Module Overview

- The DA Computational Engine (DACE) contains the implementation of the basic DA routines
- DACE Core routines: Fortran 95
  - Initialization, Memory management, error handling, DA operations
  - Each routine approximates the result of an operation by its Taylor exp.
- Interfaces: C++, MATLAB



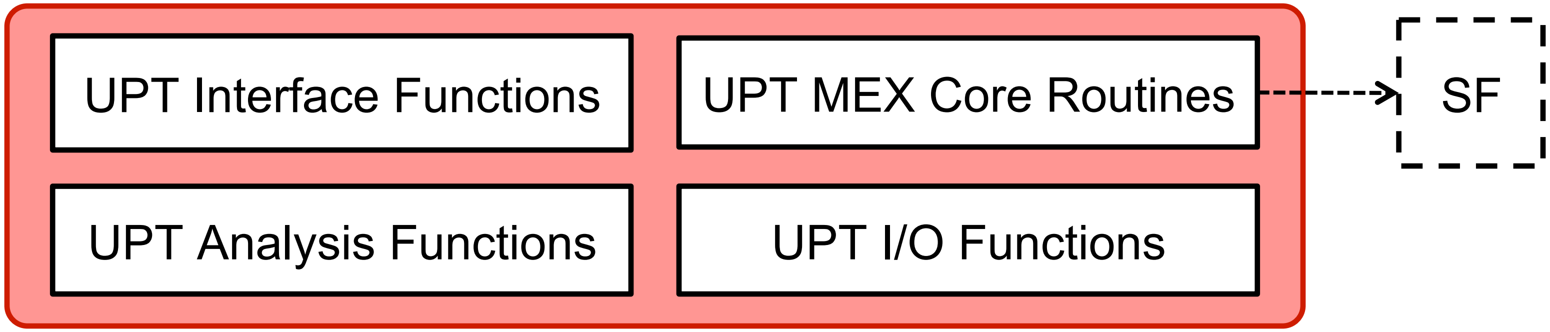
SF Module Overview

- The SF includes more advanced features:
  - DA operations between vectors and/or matrices of DA
  - DA implementation of the numerical integration schemes
  - Dynamical models and guidance models



UPT Module Overview

- The UPT is aimed at performing uncertainty propagations and statistical analyses
- The UPT comes with a set of Matlab routines to:
  - Easily interface with the SF (run DA based computations)
  - Managing simulation results
  - Easy graphical representations of the performed analyses



DIFFERENTIAL ALGEBRA SPACE TOOLBOX

Standard user

Uncertainty Propagation Tool (UPT)

Uncertainty propagation in various dynamical systems in space-related applications

Developer

Software Framework (SF)

Implementation of custom DA-based astrodynamics applications

DACE

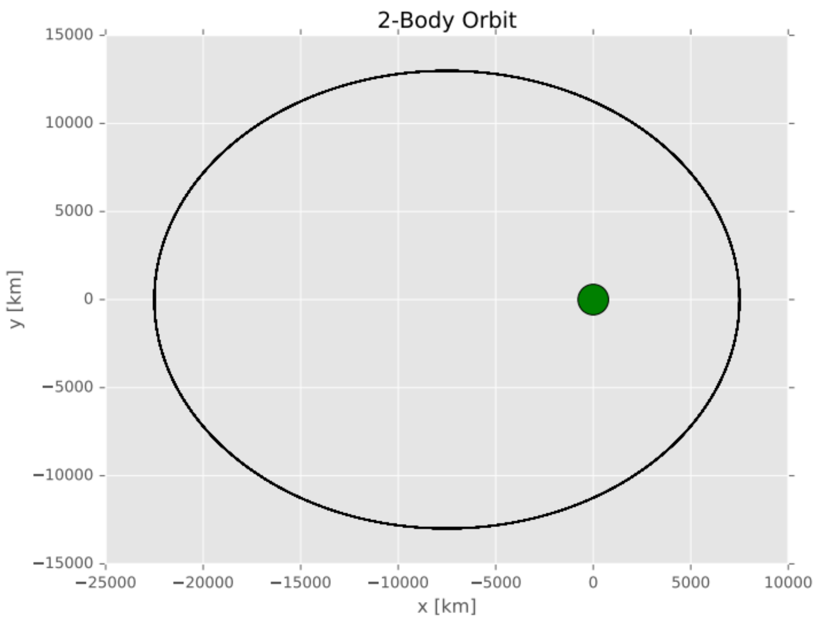
Use of DA operations for any custom application

Applications: Two-Body Dynamics

- Initial conditions

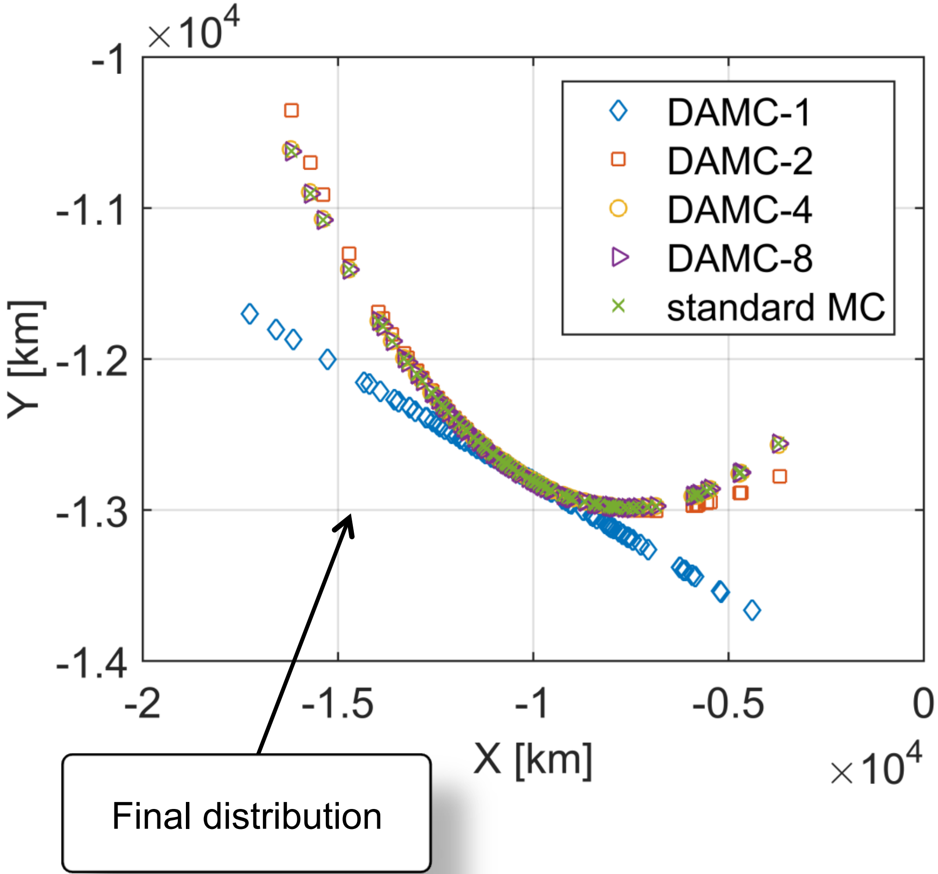
Initial Conditions			
State	Value	σ	Units
r_x	7.5e3	5e-2	km
r_y	0.0	5e-2	km
r_z	0.0	0.0	km
v_x	0.0	0.0	km/s
v_y	8.9286	0.0	km/s
v_z	0.0	0.0	km/s

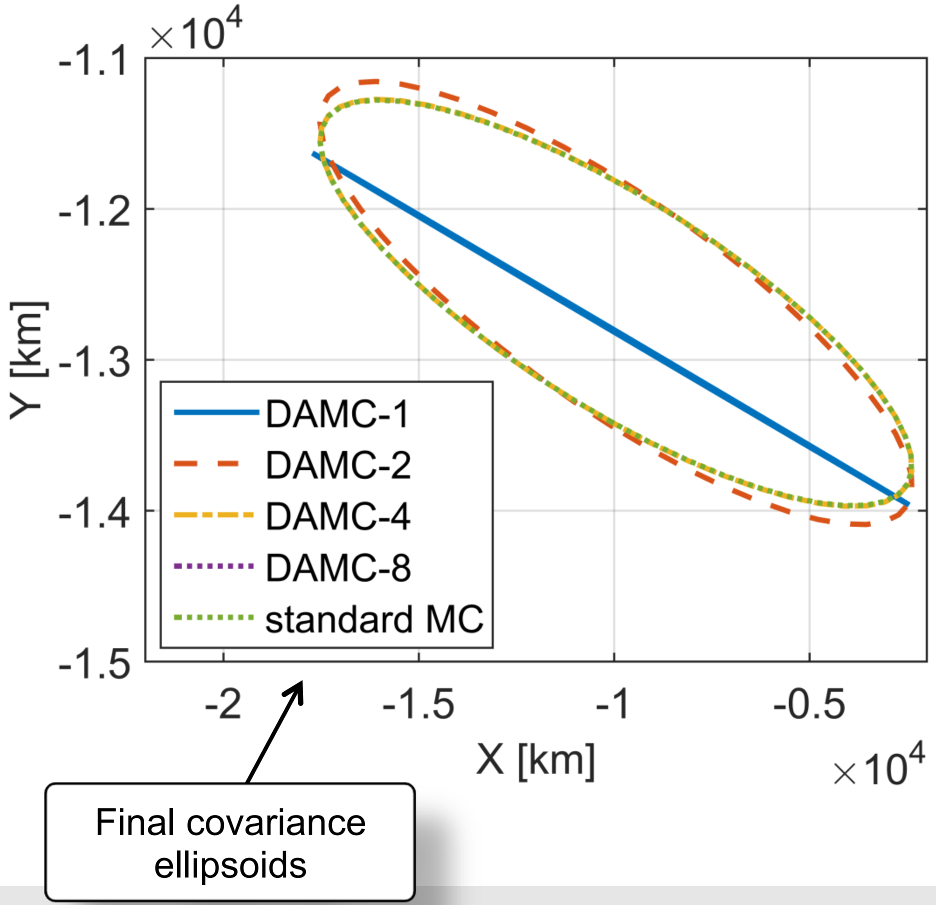
Propagation Time
$T_0 = 0.0$
T_F = 30.8 orbital periods



Applications: Two-Body Dynamics

- DA based Monte Carlo





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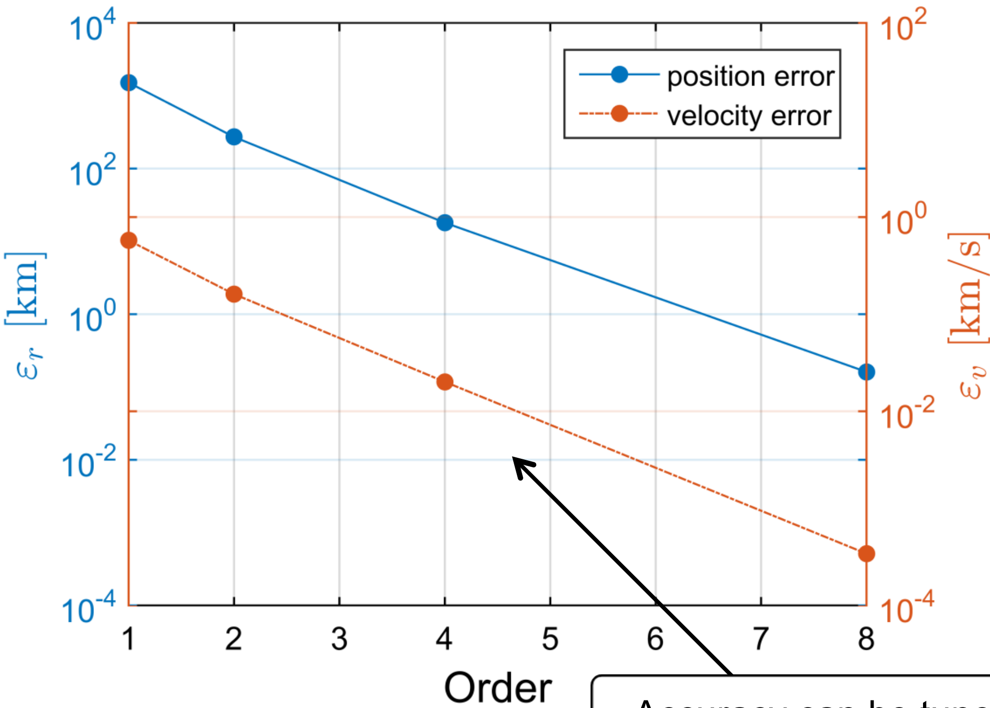
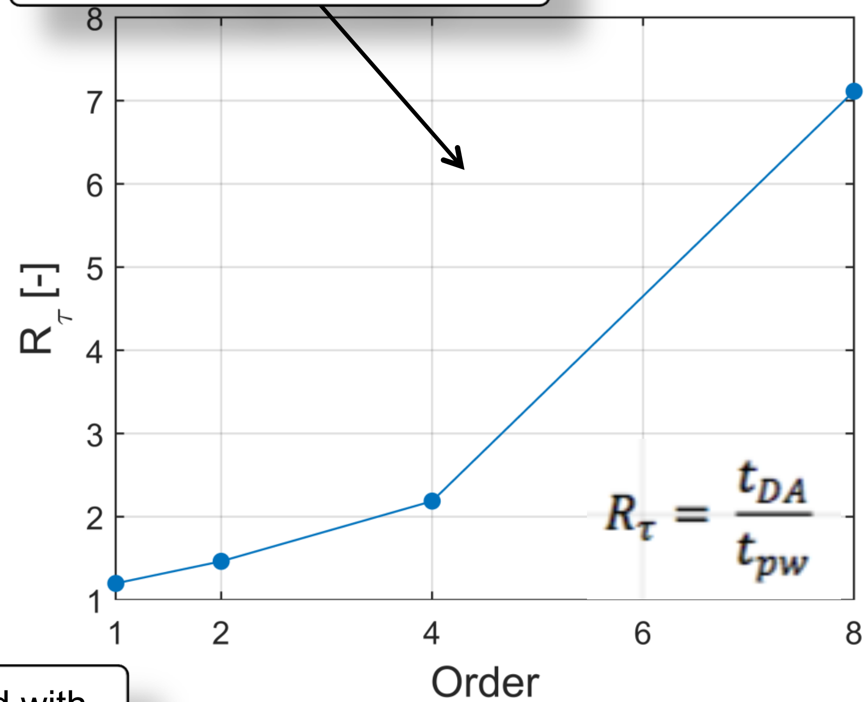


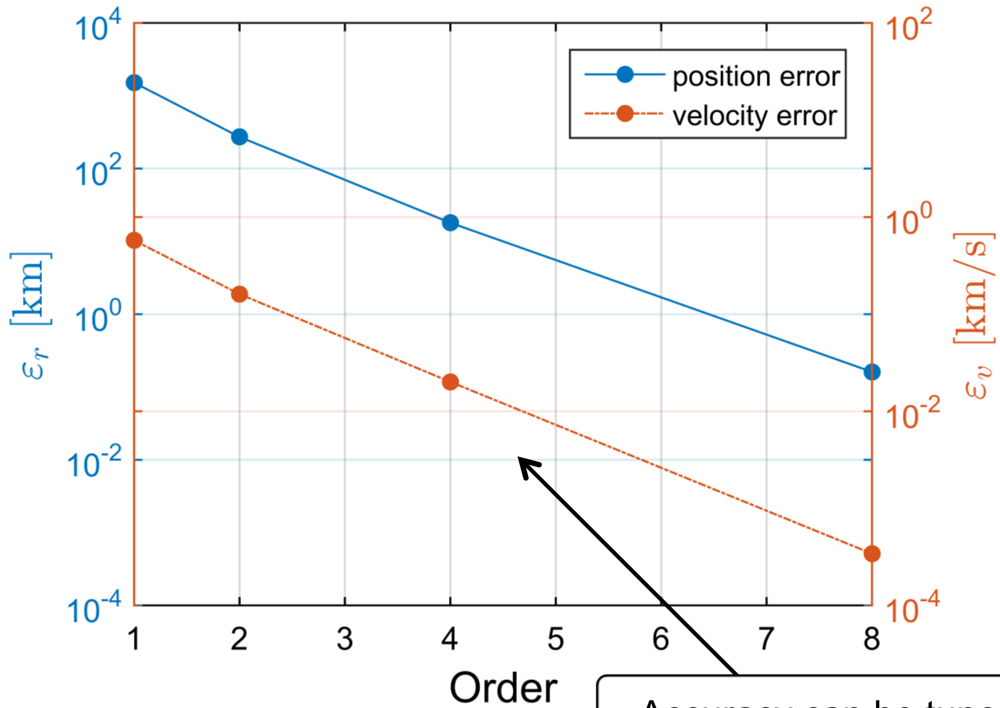


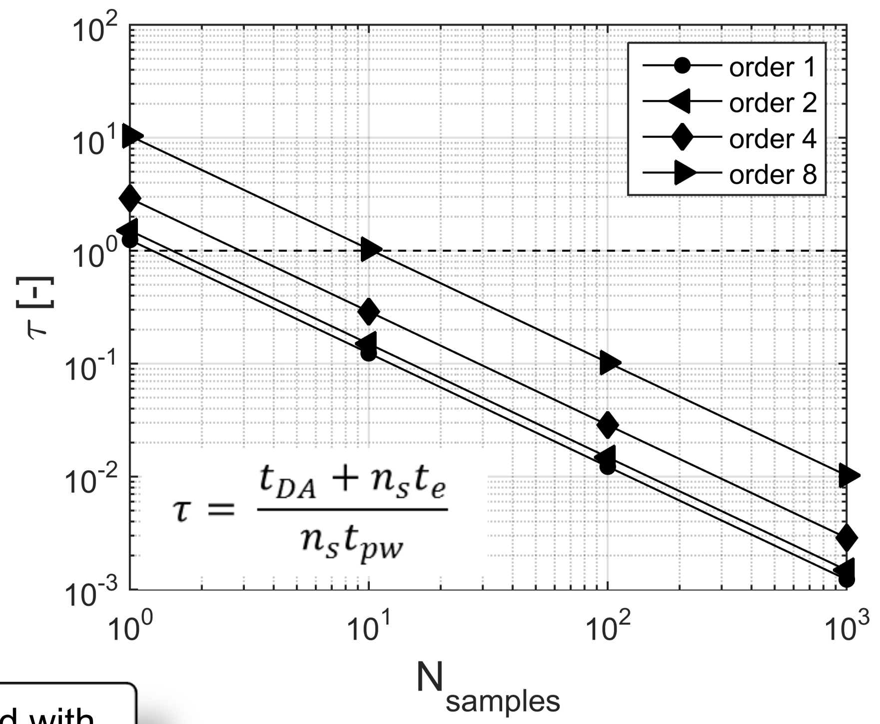
Computational cost increases for higher orders

Accuracy can be tuned with the polynomial order

APPLICATIONS: TWO-BODY DYNAMICS

- DA based Monte Carlo





Accuracy can be tuned with the polynomial order

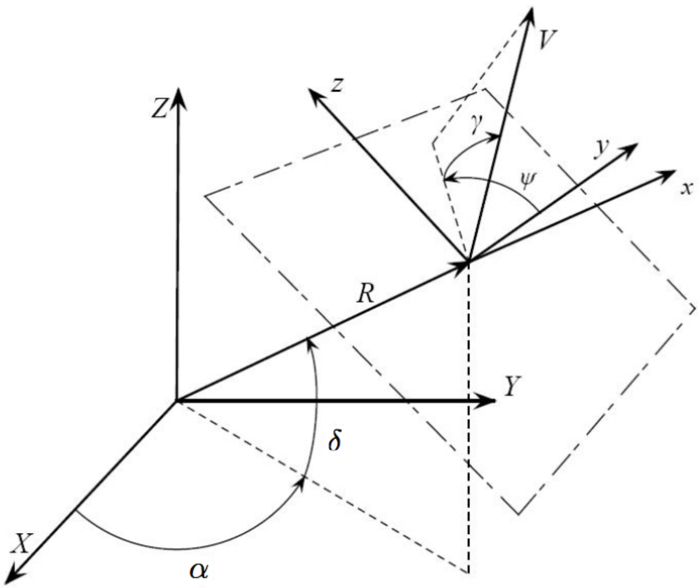
APPLICATIONS: RE-ENTRY DYNAMICS

- Re-entry dispersion analysis for a ballistic re-entry vehicle

Equations of motion

$$\begin{cases} \dot{r} &= v\sin\gamma \\ \dot{\alpha} &= \dfrac{v\cos\gamma\cos\psi}{r\cos\delta} \\ \dot{\delta} &= \dfrac{v\cos\gamma\sin\psi}{r} \\ \dot{v} &= -\dfrac{D}{m} - g\sin\gamma \\ v\dot{\gamma} &= \dfrac{L\cos\sigma}{m} - g\cos\gamma + \dfrac{v^2\cos\gamma}{r} \\ v\dot{\psi} &= \dfrac{L\sin\sigma}{m\cos\gamma} - \dfrac{v^2\tan\delta\cos\gamma\cos\psi}{r} \end{cases}$$

Reference frame



Exponential density model is used: $\rho = \rho_0 e^{-\frac{h}{\beta}}$ (NRLMSISE available)

Applications: Re-entry Dynamics

- Hayabusa re-entry*

Initial Conditions			
State	Value	σ	Units
h	201.992	0.1085	km
α	-124.28	0.0074	deg
δ	-27.33	0.008	deg
v	12.035	0.002	km/s
γ	-12.35	0.0044	deg/s
ψ	-22.06	0.0119	deg/s

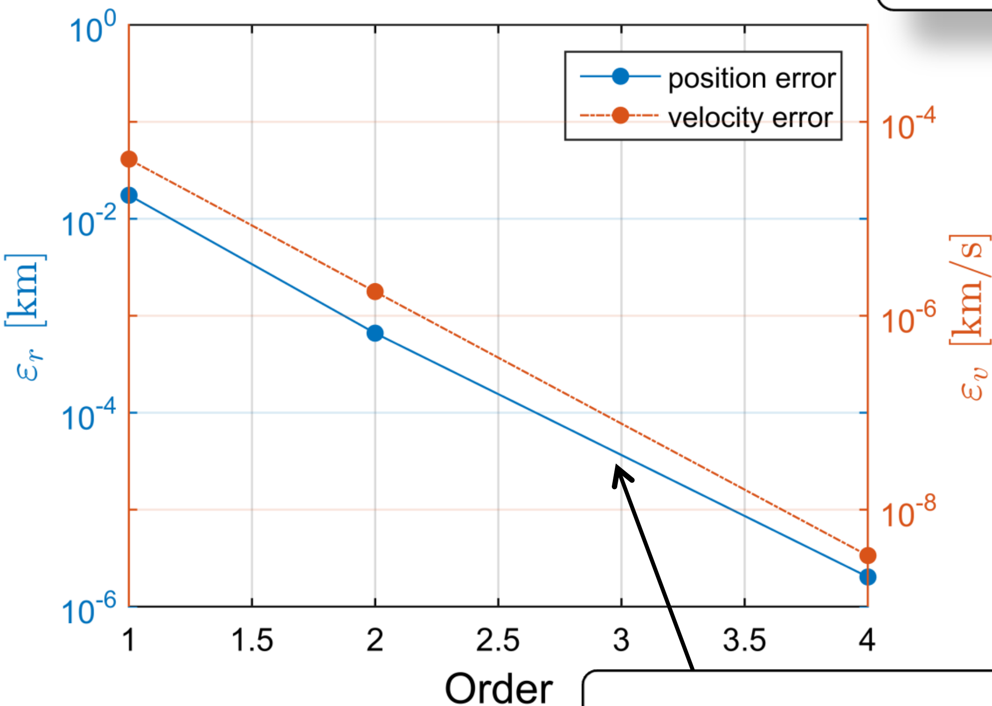
Model Parameters			
Parameter	Value	σ	Units
C_D	1.30	3.3%	-
C_L	0	-	-
m	18	-	kg
S	0.126	-	m^2
ρ_0	1.217	6.6%	kg/m^3
β	8.5	-	km

Propagation Time
T_0 = 2010-06-13, 13:51:11.47 UTC
T_F = 2010-06-13, 15:00:00.00 UTC

*Cassel et al., 2011

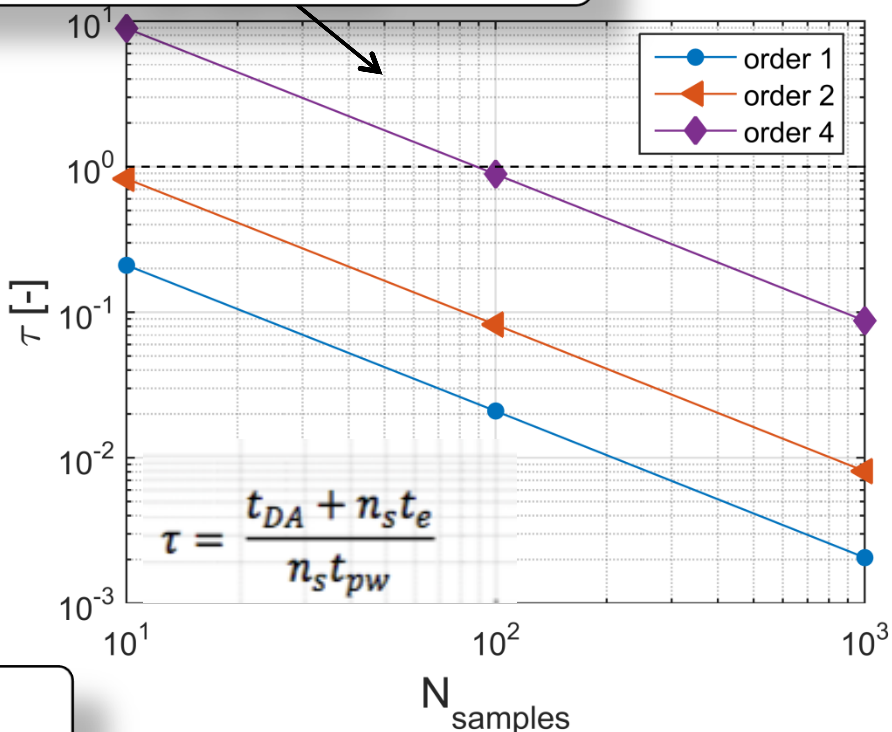
Propagation stopped at 25 km!

- DA based Monte Carlo



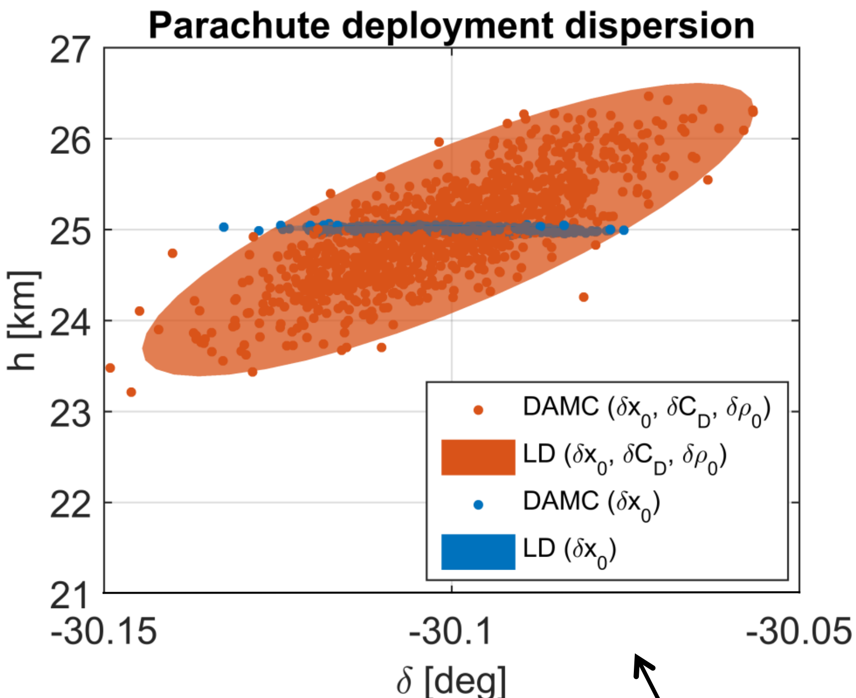
Errors decrease with higher orders

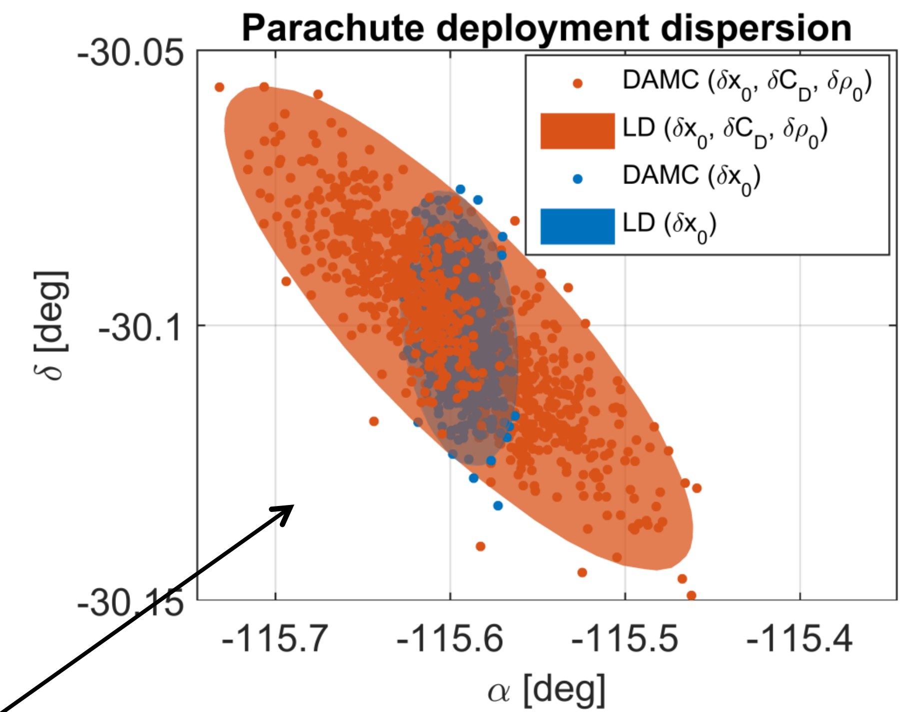
DAMC-k more efficient than classical MC for large number of samples



$$\tau = \frac{t_{DA} + n_s t_e}{n_s t_{pw}}$$

- Covariance Propagation





Uncertainty on density ρ_0 and drag coefficient C_D ➡ increased dispersion

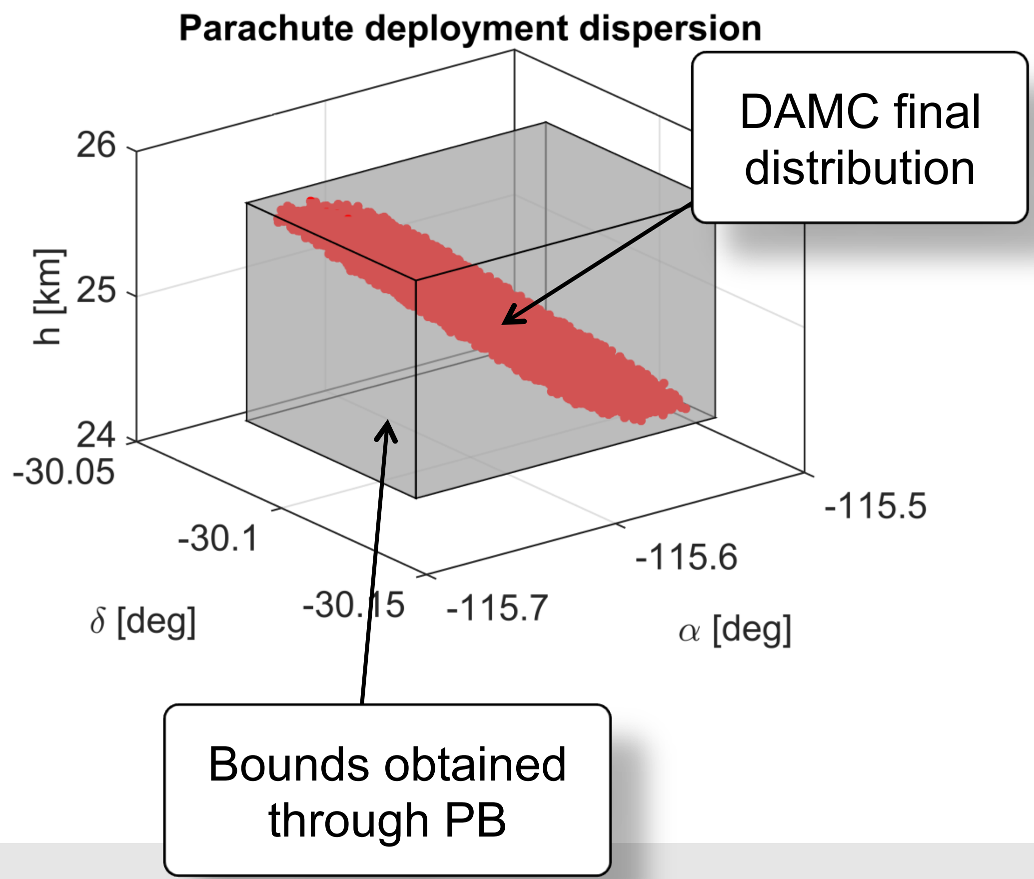
Applications: Re-entry Dynamics

- Polynomial Bounder

$$\varepsilon_{LB} = \frac{\left|\min\left(x_{f,DAMC}\right) - x_{f,LB}\right|}{x_{f,LB}}$$

$$\varepsilon_{UB} = \frac{\left|\min\left(x_{f,DAMC}\right) - x_{f,UB}\right|}{x_{f,UB}}$$

#samples	max(ε_{LB})	max(ε_{UB})
10^1	8.457e-03	4.104e-03
10^2	3.215e-03	1.979e-03
10^3	1.338e-03	8.025e-04
10^4	7.859e-04	4.634e-04
10^5	3.355e-04	4.457e-04

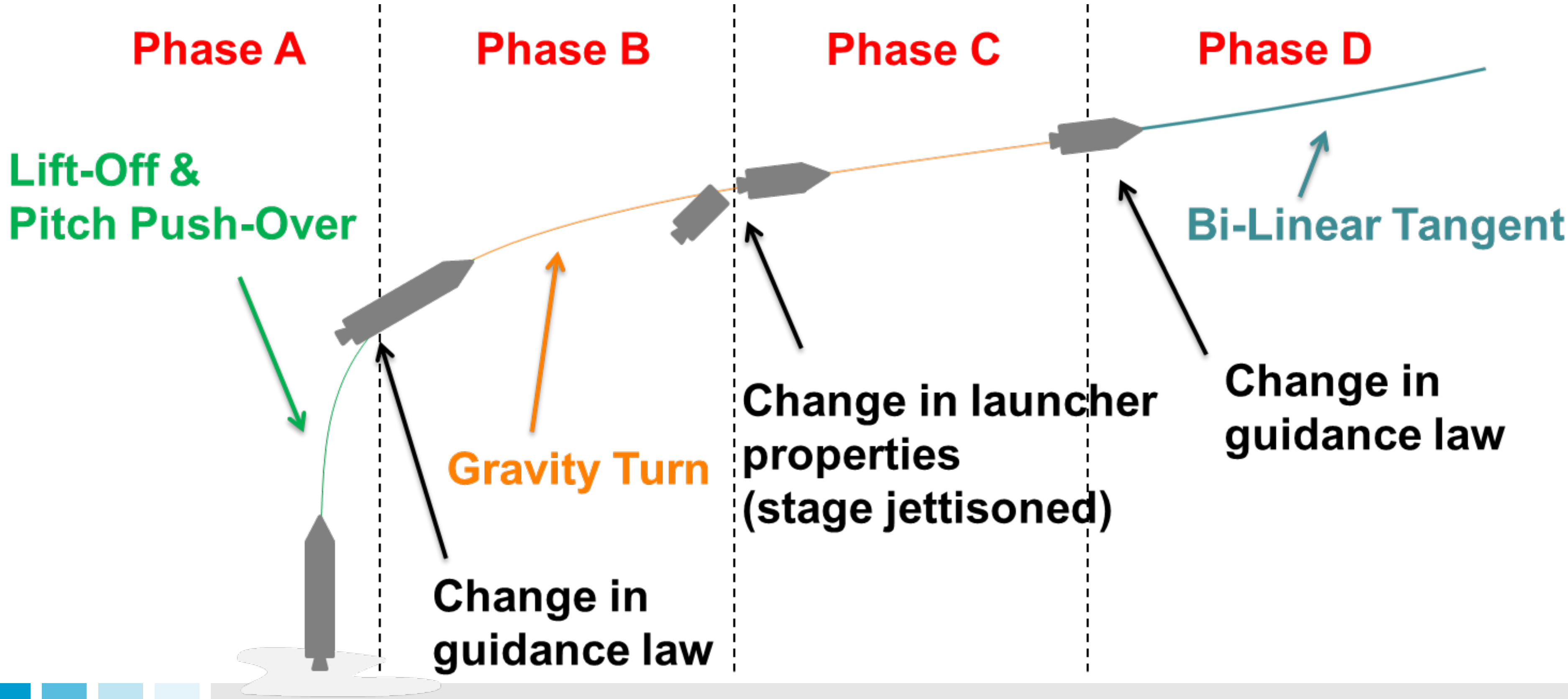


Available Dynamical Models

- The dynamical models available in the DAST are:
  - Two-Body
  - Three-Body
  - N-body
  - Re-entry
  - Relative
  - Attitude
  - Ascent
  - Rendezvous
  - Custom

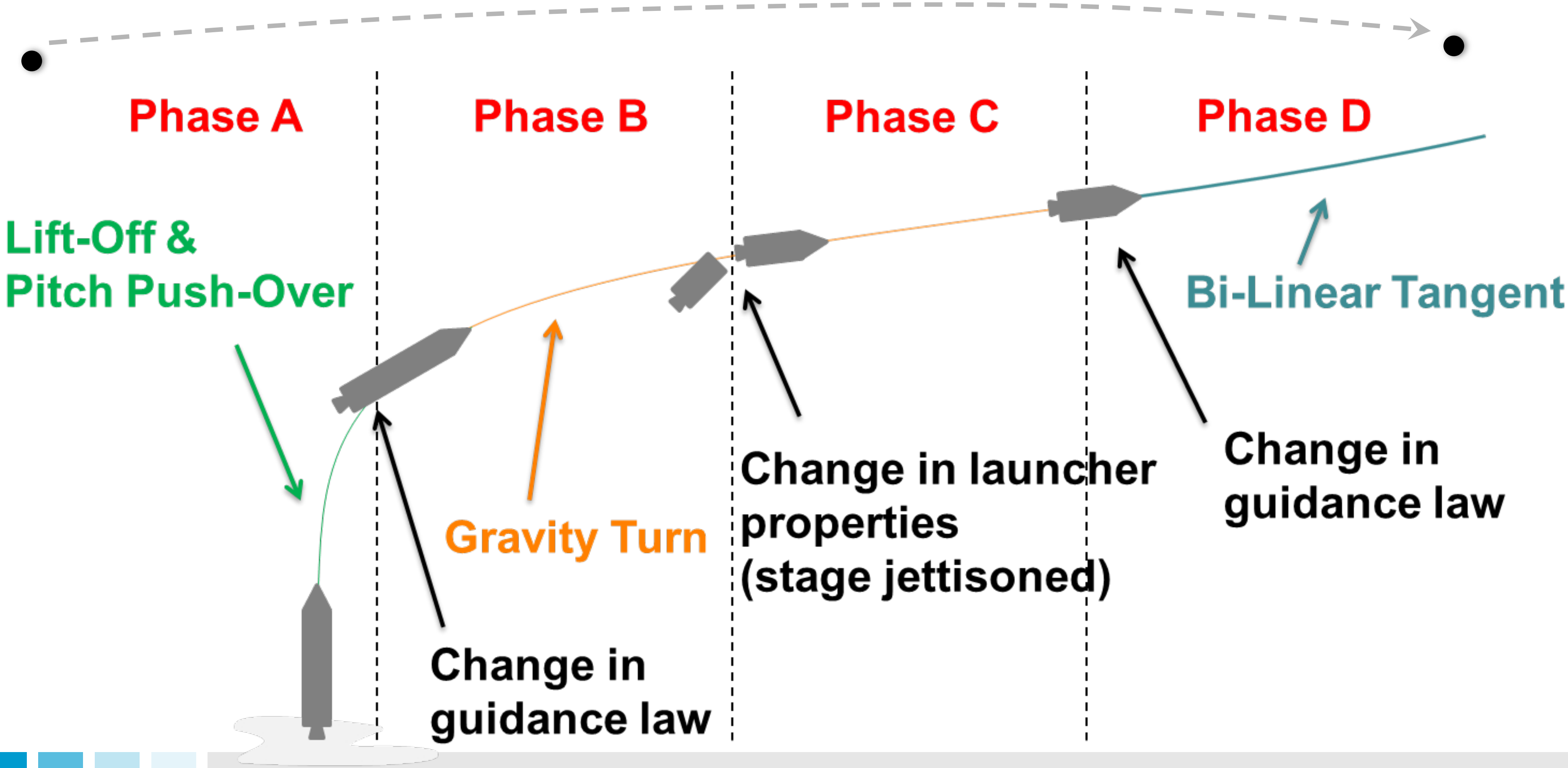
Ascent, Rendezvous → *Multi-phase approach*

MULTI-PHASE: ASCENT DYNAMICS

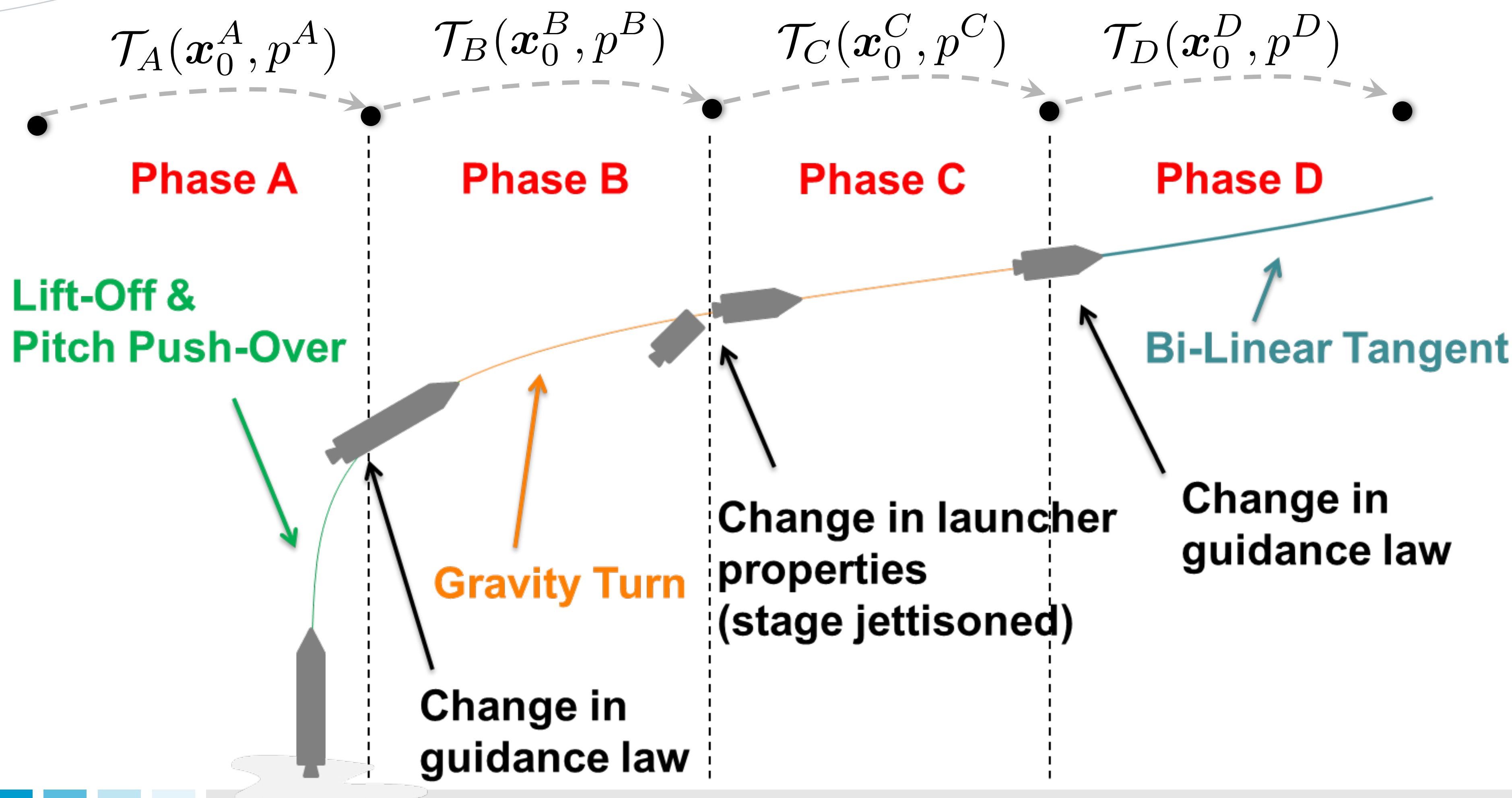


MULTI-PHASE: ASCENT DYNAMICS

$$\mathcal{T}(\boldsymbol{x}_0, p^A, p^B, p^C, p^D)$$

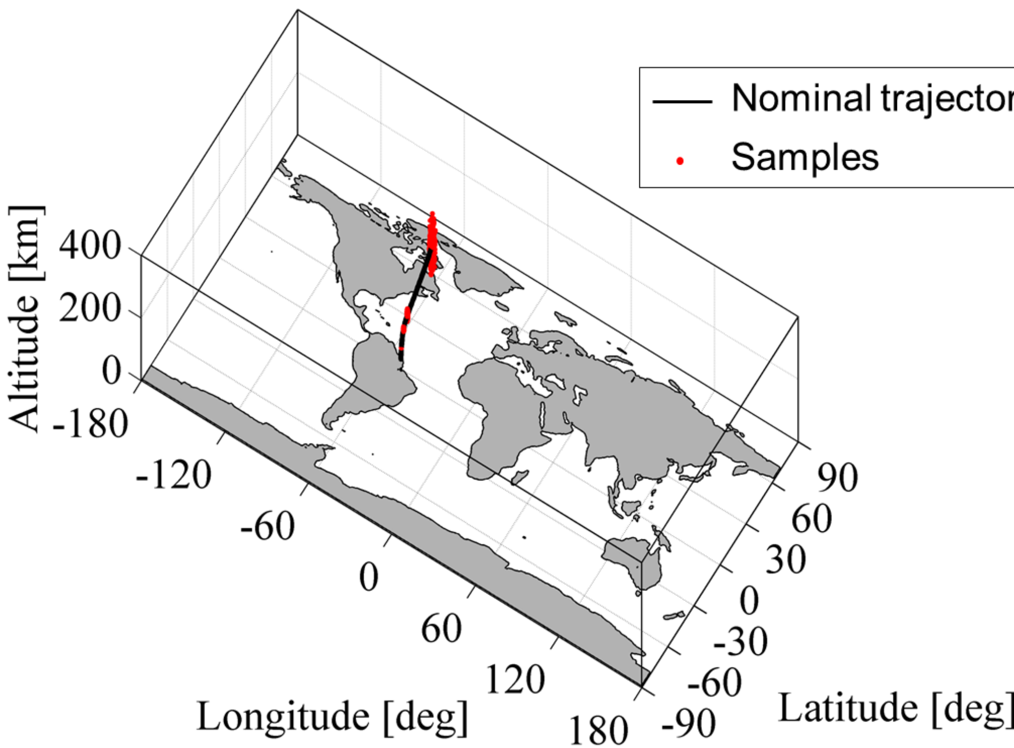


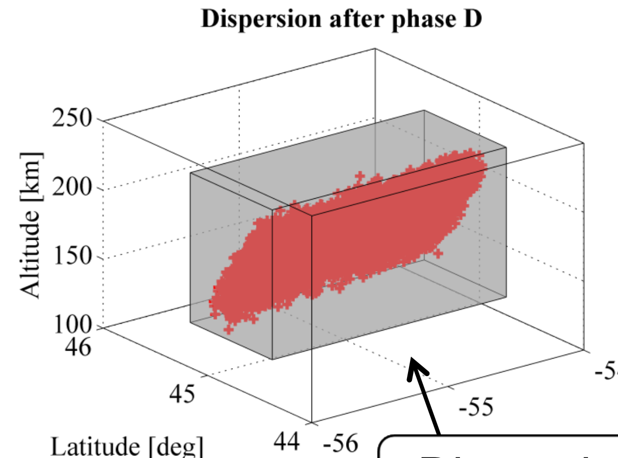
Multi-phase: Ascent Dynamics



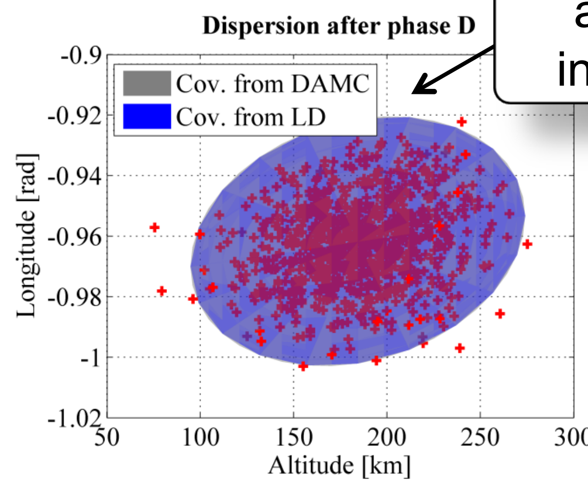
Multi-phase: Ascent Dynamics

- VEGA launcher





Dispersion at orbit insertion



CONCLUSIONS

- DAST is an efficient tool for nonlinear uncertainty propagation:
  - Propagations can be run on several ready-to-use dynamical models and any DA-compatible custom dynamical model
  - More efficient than standard Monte Carlo for typical number of samples
  - Analytical information available at the end of the propagation
- Note: method based on Taylor approximations ⇨ Size of uncertainty set and order shall guarantee sufficient accuracy
- If linear methods are sufficiently accurate for your application, you may not need to increase order, however…
- …DA relieves you from the "pain" of writing variational equations

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