In this paper, we develop a low energy transfer from the low Earth orbit (LEO) to the low lunar orbit (LLO) by employing the coupled planar restricted circular 3-body system, i.e., the Sun-Earth-spacecraft (S/C) and Earth-Moon-spacecraft (S/C) systems. We show that the departure and arrival trajectories can be chosen so that no correction maneuver is required at the patch point. To do this, we consider an optimal design problem of choosing boundary trajectories from the LEO to the LLO. Associated with energies, the family of departure trajectories from the LEO in the Sun-Earth-S/C system and the family of arrival trajectories to the LLO in the Earth-Moon-S/C system are constructed by utilizing the tube dynamics. Then, we show how the zero-maneuver trajectory can be constructed at the patch point. Finally, we compare the proposed transfer with those obtained by other methods.

**Index Terms**— Low energy transfer, boundary trajectories, tube dynamics, coupled 3-body system

### 1. INTRODUCTION

There have been many low energy transfers from the Earth to the Moon. The classical Hohmann transfer has been most commonly used for the design of orbit transfers, which is an elliptic orbit connecting with two circular orbits. However, one may need high energy for approaching to the Moon by the Hohmann transfer. In particular, under the gravitational effects due to the Sun, the Moon as well as the Earth, a low energy transfer was established by using the notion of the weak stability boundary which corresponds to stable invariant manifolds of the 4-body system (see [1]). A similar transfer was developed by coupling two planar restricted circular 3-body systems, namely, the Sun-Earth-S/C and Earth-Moon-S/C systems (see [2]). In the coupled PRC3BS, using the characteristics of the tubes, stable and unstable manifolds, an orbit near from the Earth to a patch point was constructed in the Sun-Earth-S/C system and another orbit from the vicinity of the Moon to the patch point was computed in the Earth-Moon-S/C system. Then, the orbits were connected at the patch point by some velocity correction maneuver. Recently, the idea of the coupled PRC3BS has been extended to the case of the coupled circular-elliptic 3-body system in [4] and the case in which there exist perturbations due to the Sun (or the Moon) in the context of the Bicircular system (see, [5]).

For the design point of view, one needs to construct a low energy trajectory from the boundary conditions in which the spacecraft departs from the low Earth orbit (LEO) and arrives at the low lunar orbit (LLO) are given. However, it is generally difficult to construct an optimal low energy transfer by detecting an appropriate patch point on the section. For such problems, there have been developed an optimization algorithm to find the appropriate patch point, as in [6].

In this paper, we will propose a design method for a low energy transfer from the LEO to the LLO by using the coupled PRC3BS as well as the tube dynamics. In this transfer, we will need three maneuvers; \( \Delta V_E \) from the LEO to a departure trajectory, \( \Delta V_P \) from the departure trajectory to an arrival orbit and \( \Delta V_M \) from an arrival trajectory to the LLO. First, we will make a brief review on the tube dynamics in the planar restricted circular 3-body problem (PRC3BP). Second, we will develop non-transit/transit orbits as the departure/arrival trajectories which are parametrized by energies associated with velocities of the spacecraft at the LEO/LLO. Among the families of the departure and arrival trajectories, we will establish the lowest energy transfer by choosing the departure and arrival trajectories connecting smoothly with \( \Delta V_P = 0 \).

### 2. PLANAR RESTRICTED CIRCULAR 3-BODY PROBLEM AND TUBE DYNAMICS

Consider the motion of a spacecraft under the attraction of two planets in the context of the planar restricted circular 3-body problem (PRC3BP) as in Fig.1. We assume that two bodies of large primary planets move along circular orbits with a constant angular velocity around the common mass center and also that the spacecraft with an infinitesimal mass moves in the plane of the circles. We also assume that the collision points between the spacecraft and the planets are removed. Let \( m_1 \) and \( m_2 \) be masses of the planets, respectively. The nondimensional system can be made by choosing the unit of mass as \( m_1 + m_2 \), the unit of length as the distance between the planets and the unit of time so that the planets period becomes \( 2\pi \). By this non-dimensionalization, the constant of
gravitation $G$ is set to 1. We can define the mass parameter by $\mu = m_2/(m_1 + m_2)$, and it follows that equations of motion in the rotating frame with the planets are given in local coordinates $(x, y, v_x, v_y)$ of the velocity phase space $M \approx \mathbb{R}^4$ by

$$\begin{cases}
\dot{x} = v_x, \\
\dot{y} = v_y, \\
\dot{v}_x = -\frac{(1-\mu)(x+\mu)}{r_1^2} - \frac{\mu(x-1+\mu)}{r_2^2}, \\
\dot{v}_y = -\frac{(1-\mu)y}{r_1^2} - \frac{\mu y}{r_2^2},
\end{cases}$$

where $r_1 = \sqrt{(x + \mu)^2 + y^2}$ and $r_2 = \sqrt{(x - 1 + \mu)^2 + y^2}$ denotes the distance between the spacecraft and the primary (and secondary) planet. The total energy is given by

$$E(x, y, v_x, v_y) = \frac{1}{2} (v_x^2 + v_y^2) - \frac{1}{2} (x^2 + y^2) - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2},$$

which is preserved in the PRC3BP.

As is well known, there exist five equilibrium (Lagrangian) points on the $x$ axis ($L_1, L_2, L_3$) as well as on the lines of the regular triangle ($L_4, L_5$). Fixing the energy $E$ to some value $E_0$, one can define a subset $\mathcal{E} \subset M$, called energy surface, by

$$\mathcal{E}(\mu, E_0) = \{w = (x, y, v_x, v_y) \in M \mid E(w) = E_0\}.$$  

Then, one can also define Hill’s region, where the spacecraft can move, by projecting the energy surface $\mathcal{E}$ onto the $x-y$ plane. The forbidden region is simultaneously defined as the region excluding the Hill’s region. Denoting the potential energy at a Lagrangian point by $E_{L_i}$ ($i = 1, \cdots, 5$), we choose the energy slightly greater than $E_{L_2}$, so that there is no neck region around $L_2$, as shown in Fig.2.

Since the collinear Lagrangian points ($L_1, L_2, L_3$) has the saddle $\times$ center structure, there exists the unstable orbit around the Lagrangian point, which is called the Lyapunov orbit. Further, there are stable and unstable invariant manifolds associated with the Lyapunov orbit; namely, the stable invariant manifold may tend asymptotically to the Lyapunov orbit, and the unstable invariant manifold may leave from the Lyapunov orbit as illustrated in Fig.2. These invariant manifolds are sometimes called tubes since the manifolds are homeomorphic to the cylinder $S \times \mathbb{R}$. The Hill’s region is divided into three regions by the $y$-axes including $L_1$ and $L_2$, namely, $P_1$ region including the primary planet, $P_2$ including the secondary planet, and $X$ region without $P_1$ and $P_2$ regions. Let us denote by $W_{IA}^s$ stable manifolds which tend to a Lyapunov orbit around $L_i$ ($i = 1, 2$) from $A$ ($A = P_1, P_2, X$) region and by $W_{IA}^u$ unstable manifolds which leave from a Lyapunov orbit around $L_i$ ($i = 1, 2$) toward another region $A$ ($A = P_1, P_2, X$), as illustrated in Fig.2. Now, we can see that the tubes separate orbits into transit and non-transit orbits[7]; namely, an orbit inside of the tubes is to be a transit orbit. For example, if the spacecraft is inside of the tube in some region, it is to be transported to another region. On the other hand, an orbit existing outside of the tubes is to be a non-transit orbit, where a spacecraft in a region remains in the same region.

### 3. LOW ENERGY TRANSFER

#### 3.1. Coupled PRC3BS

Now, we consider the design problem for the transfer trajectory of the spacecraft from the LEO to the LLO under the gravitational effect of the Sun, the Earth and the Moon. The Sun and the barycenter of the Earth and the Moon (Earth-Moon barycenter) rotate along the circular orbits around mass center $CM$ of the whole system, where the distance between the Sun and the Earth-Moon barycenter is $a_S (= 1.49508 \times 10^8 \text{ km})$ and the angular velocity is $\omega_S (= 1.99640 \times 10^{-7} \text{ rad/s})$. The Earth and the Moon rotate along the circular orbits around their barycenter with the angular velocity $\omega_M (= 2.66498 \times 10^{-6} \text{ rad/s})$ and the distance between the planets is $a_M (= 3.84400 \times 10^7 \text{ km})$, as shown in Fig.3. The masses of the Sun, Earth and Moon are denoted by $m_S, m_E$ and $m_M$, respectively. The spacecraft and planets move on the same plane.
Recall that the coupled PRC3BS is an approximated model of the planar bicircular restricted 4-body problem and we consider the coupled system of the Sun-Earth-S/C system with the mass parameter \( \mu_E = m_E/(m_E + m_S) = 3.02319 \times 10^{-6} \) and the Earth-Moon-S/C system with the mass parameter \( \mu_M = m_M/(m_E + m_M) = 1.21536 \times 10^{-2} \). In this paper, the expression for some quantity or set \( A \) is given by local coordinates of the Earth-Moon (E-M) rotating frame. Let \( (x, \bar{y}, \bar{v}_x, \bar{v}_y) \) denote the position and velocity of the spacecraft in the E-M rotating frame. The transformation between the S-E and E-M rotating frame is given by \( \tilde{t} = \frac{\omega_S}{\omega_M} t \).

The transformation of position is

\[
\begin{pmatrix}
\bar{x} \\
\bar{y}
\end{pmatrix} =
\begin{pmatrix}
1 - \mu_S & 0 \\
0 & \mu_M
\end{pmatrix}
+ \frac{a_M}{a_S} C(t) \begin{pmatrix} x \\ y \end{pmatrix},
\]

where \( a_S = (1 - \mu_S/\omega_S) t + \theta_{M0} \) is the angle between \( x \) axis and \( \bar{x} \) axis. The coordinate transformation for the velocity phase space, \( \mathcal{V} : M \times I \to M \times I \), is given by

\[
\begin{pmatrix}
\bar{v}_x \\
\bar{v}_y
\end{pmatrix} = \frac{a_M}{a_S \omega_S} C(t) \begin{pmatrix}
(\omega_M \bar{v}_x - (\omega_M - \omega_S) \bar{v}_y) \\
(\omega_M \bar{v}_y + (\omega_M - \omega_S) \bar{v}_x)
\end{pmatrix}.
\]

### 3.2. The Family of Departure Trajectories

For the boundary condition that the spacecraft departs from the LEO (167 km, 7.80713 km/s) by the maneuver \( \Delta V_E \), we want to get the departure trajectory in the Sun-Earth-S/C system. Set the departure point on the LEO in the S-E rotating frame as \( (\bar{x}, \bar{y}, \bar{v}_x, \bar{v}_y) = (1 - \mu_S - r_{LEO}, 0, 0, -\bar{v}_{LEO}) = (0.999953, 0, 0, -0.261364) \), where \( r_{LEO} = 4.37038 \times 10^{-5} \) denotes the altitude of the spacecraft from the Earth’s center and \( \bar{v}_{LEO} = 0.261364 \) the velocity of the LEO in the S-E rotating frame. The energy at the point is \( E^{SE}_{LEO} = -1.53501 \). It is assumed that \( \Delta V_E \) is produced in parallel to the velocity of the LEO in Fig. 4. Hence the spacecraft is transferred to the patch point \( \bar{w}_p = (x_D, y_D, v_{xD}, v_{yD}) = (1 - \mu_S - r_{LEO}, 0, 0, -\bar{v}_{LEO} - \Delta V_E) = (0.999953, 0, 0, -\bar{v}_{LEO} - \Delta V_E) \). Note that the energy \( E^{SE} \) after the maneuver is uniquely obtained by \( \Delta V_E \).

Here, we construct the non-transit orbit from the patch point as the departure trajectory in order to remain in the Earth region. So, the patch point after the maneuver is required to be outside of the stable manifold \( W^{s}_{2,SE} \). Let us find out an energy range (associated with \( \Delta V_E \)) satisfying this condition. To do this, set the Poincaré section \( \mathcal{U} \) by

\[
\mathcal{U} = \{(\bar{x}, \bar{y}, \bar{v}_x, \bar{v}_y) \in \mathcal{E}(\mu_S, E^{SE}_0) \mid \bar{x} < 1 - \mu_S, \bar{y} = 0, \bar{v}_x > 0 \}.
\]

Set the energy at Lagrangian point \( \tilde{L}_2 \), \( (\bar{x}, \bar{y}, \bar{v}_x, \bar{v}_y) = (0.999997, 0, 0, 0) \), by \( E^{SE}_{L_2} = -1.50045 \), and there exists the neck region within the energy range \( E^{SE}_{L_2} < E^{SE} \). Now, let us find out the upper limit of the departure trajectory for this energy region by investigating the stable manifold \( W^{s}_{2,SE} \) on \( \mathcal{U} \).

The LEO and the patch point are illustrated together with \( W^{s}_{2,SE} \cap \mathcal{U} \) for the case of \( E^{SE}_0 = -1.50040 \) in Fig. 5. One
can detect that the energy range is $E_{L_2}^{SE} \leq E_0^{SE} \leq E_{D_{max}} = -1.50040$ where an orbit is to be a non-transit orbit. In particular, the stable manifold $W_{2,E}^{s}$ with $E_0^{SE} = E_{D_{max}}$ is illustrated in Fig.6. Needless to say, for the region $E_0^{SE} < E_{L_2}^{SE}$, the spacecraft is transported to the exterior region since the patch point is inside of $W_{2,E}^{s}$.

Fig. 5: Boundary of non-transit orbits
($E_0^{SE} = -1.50040$)

Fig. 6: Stable manifolds $W_{2,E}^{s}$
($E_{D_{max}} = -1.50040$)

In this way, we can define the family of the non-transit orbits parametrized by the energy $E_0^{SE}$ as

$$\mathcal{D}(E_0^{SE}) = \{\phi_t(w_D) \in M \mid w_D \in E(M, E_0^{SE}),
E_0^{SE} \in [E_{L_2}^{SE}, E_{D_{max}}^{SE}]\}.$$ 

In the above, $\phi_t : M \rightarrow M$ is the flow, where $t \in I \subset \mathbb{R}$. Associated with 50 values for the energy $E_0^{SE} \in [E_{L_2}^{SE}, E_{D_{max}}^{SE}]$, the family of the non-transit orbits is illustrated in Fig.7.

3.3. The Family of Arrival Trajectories

One needs to consider another boundary condition for the spacecraft; namely, the spacecraft is to be transferred from an arrival trajectory into the LLO (100 km, 1.65346 km/s) by a correction maneuver $\Delta V_M$ in the Earth-Moon-spacecraft system. Finally, the spacecraft is to arrive at $(x, y, v_x, v_y) = (1 - \mu_M - r_{LLO}, 0, 0, -v_{LLO}) = (0.983066, 0, 0, -1.58974)$, where the energy at the point is $E_{LLO} = -2.75466$, $r_{LLO} = 4.78018 \times 10^{-3}$ indicates the altitude of the spacecraft from the Earth’s center and $v_{LLO} = 1.58974$ the velocity of the spacecraft at the LLO in the E-M rotating frame. We assume the correction maneuver is produced in parallel to the velocity of the LLO in Fig.8. One can determine the patch point as $w_A = (x_A, y_A, v_x, v_y) = (1 - \mu_M - r_{LLO}, 0, 0, -v_{LLO} - \Delta V_M) = (0.983066, 0, 0, -v_{LLO} - \Delta V_M)$. Since the energy at the LLO is given, i.e., $E_{LLO} = -2.75466$, one can uniquely determine the energy $E_0^{EM}$ at the patch point associated with some correction maneuver $\Delta V_M$.

Here, we construct the transit orbit from the LLO to the exterior region in the backward time as the arrival trajectory. The patch point is so chosen that it is to be inside of the unstable manifold $W_{2,M}^{u}$. We shall determine the energy range (associated with $\Delta V_M$) to obtain the transit orbit and set the Poincaré section $\mathcal{U}$ by

$$\mathcal{U} = \{(x, y, v_x, v_y) \in E(M, E_0^{EM}) \mid x < 1 - \mu_M,
y = 0, v_x < 0\}.$$ 

Since the energy at the Lagrangian point $L_3$ is $E_{L_3} = -1.50608$, where $L_3$ is $(x, y, v_x, v_y) = (-1.00506, 0, 0, 0)$,
the upper limit of the energy is given by $E_{0}^{EM} \leq E_{L3}$. In this energy range, we illustrate the unstable manifold $W^{u}_{2M}$ on $U$, namely, the subset $W^{u}_{2M} \cap U$ for the case of $E_{0}^{EM} = -1.57961$ in Fig.9. So, we can determine the lower limit of the energy as $E_{v_{0}}^{EM} = -1.57961 \leq E_{0}^{EM} \leq E_{L3}^{EM}$. We illustrate $W^{u}_{2M}$ with $E_{0}^{EM} = E_{v_{0}}^{EM}$ in Fig.10.

![Fig. 9: Boundary of non-transit orbits ($E_{0}^{EM} = -1.57961$)](image)

One can obtain the family of the transit orbits parametrized by the energy $E_{0}^{EM}$, which is given by

$A(E_{0}^{EM}) = \{ \phi_{-t}(w_{A}) \in M \mid w_{A} \in E_{E}, E_{0}^{EM} \in [E_{v_{0}}^{EM}, E_{v_{0}}^{EM}].

We show the family of the transit orbits associated with 50 values of $E_{0}^{EM}$ in Fig.11.

![Fig. 10: Stable manifold $W^{v}_{2M}$ ($E_{v_{0}}^{EM} = -1.57961$)](image)

3.4. Design of the Transfer Orbit from the LEO to the LLO

In this section, we show how the transfer orbit from the LEO to the LLO can be obtained by patching the departure and arrival trajectories in the previous sections. In particular, we demonstrate that the optimal transfer can be made in the sense that $\Delta V_{P} = 0$, though the maneuver $\Delta V_{P}$ is generally required in the patching.

![Fig. 11: Family of transit orbits to LLO ($E_{0}^{EM} \in [E_{v_{0}}^{EM}, E_{L3}^{EM}])$](image)

Let us consider the Sun-Earth-S/C and Earth-Moon-S/C systems in the S-E rotating frame in order to patch the trajectories. The Earth-Moon-S/C system in the S-E rotating frame can be considered to be a non-autonomous system since it depends on the angle $\theta_{M} \in [0, 2\pi]$. We show the arrival trajectories $\varphi(A(E_{0}^{EM}))$ at $\theta_{M} = 2.58030$ in addition to the family of the departure trajectories in Fig.12.

![Fig. 12: Departure and arrival trajectories in the S-E rotating coordinate ($\theta_{M} = 2.58030$)](image)

To determine a patch point, set the Poincaré section on the $x$ axis by

$\bar{U} = \{ (x, y, v_{x}, v_{y}) \mid x > 1 - \mu_{S}, y = 0, v_{y} > 0 \}.$

We show the family of the departure trajectories crossing the section $\bar{U}$ given by $D(E_{0}^{SE}) \cap \bar{U}$ and the subset of the family of the arrival trajectories $\varphi(A(E_{0}^{EM})) \cap \bar{U}$ for $\theta_{M} \in [0, 2\pi]$ in Fig.13. Thus, the patch point can be uniquely chosen at the intersection of $D(E_{0}^{SE}) \cap \bar{U}$ and $\varphi(A(E_{0}^{EM})) \cap \bar{U}$ as

$(x_{P}, y_{P}, v_{xP}, v_{yP}) = (1.00521, 0, -0.0105875, 0.0173207).$

The transfer trajectory is obtained by smoothly connecting the departure and arrival trajectories, where the trajectory
is optimal in the sense that the maneuver $\Delta V_P$ is zero. The departure trajectory can be computed from the initial point as

$$(x_D, y_D, v_{xD}, v_{yD}) = (0.999953, 0, 0, -0.370838),$$

where the energy is $E_{SE0} = 1.50037$. It follows that $V_E = 3.270 \text{ km/s}$. The final point of the arrival trajectory in the S-E rotating frame is given by

$$(x_A, y_A, v_{xA}, v_{yA}) = (0.997859, 0.00134456, 0.0238594, 0.0379475),$$

and the angle at the final point is $\theta_M = 2.58030$. The energy of the arrival trajectory is $E_{EM0} = 1.56243$ and hence the maneuver is $\Delta V_M = 0.642 \text{ km/s}$.

As in Fig. 14, it is clear that the patch point is outside of the unstable manifold $W_{2,E}^u$ of the Sun-Earth-S/C system and inside of the stable manifold $W_{2,M}^s$ of the Earth-Moon-S/C system on $\bar{U}$.

Finally, we compare the proposed transfer with the Hohmann transfer and a transfer obtained by the previous coupled PRC3BS [2] in Table 1. We obtain the Hohmann transfer as the elliptic orbit connecting with the LEO and the lunar orbit. Using the previous approach of the coupled PRC3BS, the energies of the Sun-Earth-S/C and E-M-S/C systems are $E_{SE0}^{SE} = -1.50037$ and $E_{EM0}^{EM} = -1.58340$ respectively. As in Table 1, it follows that the total maneuver of the proposed approach is $0.067 \text{ km/s}$ fewer than the Hohmann transfer. In addition, the proposed transfer does not require any maneuver in patching comparing with the previous coupled PRC3BS.

### Table 1: Maneuver ($\Delta V \text{ [km/s]}$) and flight time ($T \text{ [d]}$)

<table>
<thead>
<tr>
<th>Transfer</th>
<th>$\Delta V_E$</th>
<th>$\Delta V_M$</th>
<th>$\Delta V_P$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hohmann</td>
<td>3.141</td>
<td>0.838</td>
<td>–</td>
<td>5</td>
</tr>
<tr>
<td>Coupled PRC3BS[2]</td>
<td>3.537</td>
<td>1.989</td>
<td>0.998</td>
<td>179</td>
</tr>
<tr>
<td>Proposed approach</td>
<td>3.270</td>
<td>0.642</td>
<td>0</td>
<td>102</td>
</tr>
</tbody>
</table>

Fig. 13: Patch point and the departure and arrival trajectories ($\theta_M \in [0, 2\pi]$) on $\bar{U}$

Fig. 14: Patch point and invariant manifolds on $\bar{U}$

Fig. 15: Transfer from the LEO to the LLO in the SE rotating coordinate

Fig. 16: Transfer from the LEO to the LLO in the EM rotating coordinate
4. CONCLUSION

In this paper, we have shown design of a low energy transfer under some given boundary conditions where the spacecraft departs from a low Earth orbit (LEO) and arrives at a low lunar orbit (LLO). The transfer has been developed by using the coupled planar restricted circular 3-body system, namely, the Sun-Earth-spacecraft and Earth-Moon-spacecraft systems. First, we have constructed the family of the departure trajectories so that the trajectories are to be the non-transit orbits from the LEO existed outside of the stable manifold in the Sun-Earth-spacecraft system. The family of the arrival trajectories has been obtained by the transit orbits inside of the unstable manifold in the Earth-Moon-spacecraft system. By parametrizing the family of the arrival trajectories by the angle $\theta_M$ in the Sun-Earth rotating frame, we have obtained the optimized patch point at which the departure and arrival trajectories are connected smoothly with $\Delta V_p = 0$. Finally, we have illustrated the low energy transfer from the LEO to the LLO and the validity of the proposed approach in comparison with other approaches.

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6. REFERENCES


