TUBE DYNAMICS AND LOW ENERGY TRAJECTORY FROM THE EARTH TO THE MOON IN THE COUPLED THREE-BODY SYSTEM

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ABSTRACT

In this paper, we develop a low energy transfer from the low Earth orbit (LEO) to the low lunar orbit (LLO) by employing the coupled planar restricted circular 3-body system, i.e., the Sun-Earth-spacecraft (S/C) and Earth-Moon-spacecraft (S/C) systems. We show that the departure and arrival trajectories can be chosen so that no correction maneuver is required at the patch point. To do this, we consider an optimal design problem of choosing boundary trajectories from the LEO to the LLO. Associated with energies, the family of departure trajectories from the LEO in the Sun-Earth-S/C system and the family of arrival trajectories to the LLO in the Earth-Moon-S/C system are constructed by utilizing the tube dynamics. Then, we show how the zero-maneuver trajectory can be constructed at the patch point. Finally, we compare the proposed transfer with those obtained by other methods.

Index Terms— Low energy transfer, boundary trajectories, tube dynamics, coupled 3-body system

1. INTRODUCTION

There have been many low energy transfers from the Earth to the Moon. The classical Hohmann transfer has been most commonly used for the design of orbit transfers, which is an elliptic orbit connecting with two circular orbits. However, one may need high energy for approaching to the Moon by the Hohmann transfer. In particular, under the gravitational effects due to the Sun, the Moon as well as the Earth, a low energy transfer was established by using the notion of the *weak stability boundary* which corresponds to stable invariant manifolds of the 4-body system (see [1]). A similar transfer was developed by *coupling two planar restricted circular 3-body systems*, namely, the Sun-Earth-S/C and Earth-Moon-S/C systems (see [2]). In the coupled PRC3BS, using the characteristics of the tubes, stable and unstable manifolds, an orbit near from the Earth to a patch point was constructed in the Sun-Earth-S/C system and another orbit from the vicinity of the Moon to the patch point was computed in the Earth-Moon-S/C system. Then, the orbits were connected at the patch point by some velocity correction maneuver. Recently, the idea of the coupled PRC3BS has been extended to the case of the coupled circular-elliptic 3-body system in [4] and the case in which there exist perturbations due to the Sun (or the Moon) in the context of the Bicircular system (see, [5]).

For the design point of view, one needs to construct a low energy trajectory from the boundary conditions in which the spacecraft departs from the low Earth orbit (LEO) and arrives at the low lunar orbit (LLO) are given. However, it is generally difficult to construct an optimal low energy transfer by detecting an appropriate patch point on the section. For such problems, there have been developed an optimization algorithm to find the appropriate patch point, as in [6].

In this paper, we will propose a design method for a low energy transfer from the LEO to the LLO by using the coupled PRC3BS as well as the tube dynamics. In this transfer, we will need three maneuvers; ΔV_{E} from the LEO to a departure trajectory, ΔV_{P} from the departure trajectory to an arrival orbit and ΔV_{M} from an arrival trajectory to the LLO. First, we will make a brief review on the tube dynamics in the planar restricted circular 3-body problem (PRC3BP). Second, we will develop non-transit/transit orbits as the departure/arrival trajectories which are parametrized by energies associated with velocities of the spacecraft at the LEO/LLO. Among the families of the departure and arrival trajectories, we will establish the lowest energy transfer by choosing the departure and arrival trajectories connecting smoothly with $\Delta V_{\mathrm{P}} = 0$.

2. PLANAR RESTRICTED CIRCULAR 3-BODY PROBLEM AND TUBE DYNAMICS

Consider the motion of a spacecraft under the attraction of two planets in the context of the planar restricted circular 3-body problem (PRC3BP) as in Fig.1. We assume that two bodies of large primary planets move along circular orbits with a constant angular velocity around the common mass center and also that the spacecraft with an infinitesimal mass moves in the plane of the circles. We also assume that the collision points between the spacecraft and the planets are removed. Let m_1 and m_2 be masses of the planets, respectively. The nondimensional system can be made by choosing the unit of mass as $m_1 + m_2$, the unit of length as the distance between the planets and the unit of time so that the planets period becomes 2π. By this non-dimensionalization, the constant of

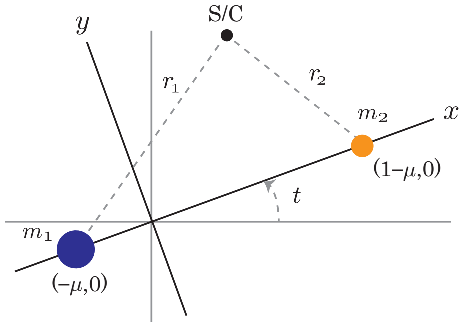

Fig. 1: PRC3BP

gravitation G is set to 1. We can define the mass parameter by $\mu = m_2/(m_1+m_2)$, and it follows that equations of motion in the rotating frame with the planets are given in local coordinates (x, y, v_x, v_y) of the velocity phase space $M \approx \mathbb{R}^4$ by

$$\begin{cases} \dot{x} = v_x, \\ \dot{y} = v_y, \\ \dot{v}_x - 2v_y = x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}, \\ \dot{v}_y + 2v_x = y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}, \end{cases} \quad (1)$$

where $r_1 = \sqrt{(x+\mu)^2+y^2}$ (and $r_2 = \sqrt{(x-1+\mu)^2+y^2}$) denotes the distance between the spacecraft and the primary (and secondary) planet. The total energy is given by

$$E(x, y, v_x, v_y) = \frac{1}{2}(v_x^2+v_y^2) - \frac{1}{2}(x^2+y^2) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}, \quad (2)$$

which is preserved in the PRC3BP.

As is well known, there exist five equilibrium (Lagrangian) points on the x axis (L_1, L_2, L_3) as well as on the lines of the regular triangle (L_4, L_5). Fixing the energy E to some value E_0, one can define a subset $\mathcal{E} \subset M$, called *energy surface*, by

$$\mathcal{E}(\mu, E_0) = \{w = (x, y, v_x, v_y) \in M \mid E(w) = E_0\}. \quad (3)$$

Then, one can also define Hill's region, where the spacecraft can move, by projecting the energy surface $\mathcal{E}$ onto the $x-y$ plane. The forbidden region is simultaneously defined as the region excluding the Hill's region. Denoting the potential energy at a Lagrangian point by E_{L_i} $(i = 1, \cdots, 5)$, we choose the energy slightly greater than E_{L_2} so that there exist neck region around L_2, as shown in Fig.2.

Since the collinear Lagrangian points (L_1, L_2, L_3) has the $saddle \times center$ structure, there exists the unstable orbit around the Lagrangian point, which is called the *Lyapunov orbit*. Further, there are stable and unstable invariant manifolds associated with the Lyapunov orbit; namely, the stable invariant manifold may tend asymptotically to the Lyapunov orbit, and the unstable invariant manifold may leave from

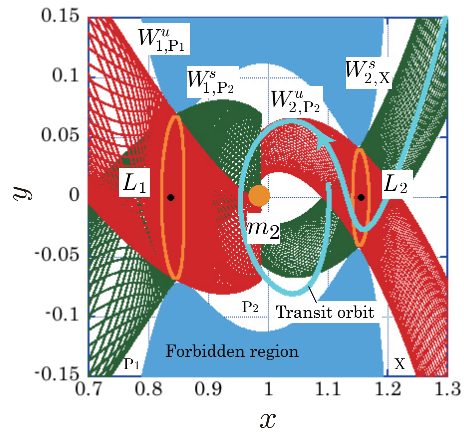

Fig. 2: Flows near the secondary planet

the Lyapunov orbit as illustrated in Fig.2. These invariant manifolds are sometimes called *tubes* since the manifolds are homeomorphic to the cylinder $S \times \mathbb{R}$. The Hill's region is divided into three regions by the y-axes including L_1 and L_2, namely, P_1 region including the primary planet, P_2 including the secondary planet, and X region without P_1 and P_2 regions. So let us denote by $W^s_{i,\mathrm{A}}$ stable manifolds which tend to a Lyapunov orbit around L_i $(i = 1, 2)$ from A $(\mathrm{A} = \mathrm{P}_1, \mathrm{P}_2, \mathrm{X})$ region and by $W^u_{i,\mathrm{A}}$ unstable manifolds which leave from a Lyapunov orbit around L_i $(i = 1, 2)$ toward another region A $(\mathrm{A} = \mathrm{P}_1, \mathrm{P}_2, \mathrm{X})$, as illustrated in Fig.2. Now, we can see that the tubes separate orbits into *transit* and *non-transit orbits*[7]; namely, an orbit inside of the tubes is to be a transit orbit. For example, if the spacecraft is inside of the tube in some region, it is to be transported to another region. On the other hand, an orbit existing outside of the tubes is to be a non-transit orbit, where a spacecraft in a region remains in the same region.

3. LOW ENERGY TRANSFER

3.1. Coupled PRC3BS

Now, we consider the design problem for the transfer trajectory of the spacecraft from the LEO to the LLO under the gravitational effect of the Sun, the Earth and the Moon. The Sun and the barycenter of the Earth and the Moon (Earth-Moon barycenter) rotate along the circular orbits around mass center CM of the whole system, where the distance between the Sun and the Earth-Moon barycenter is $a_\mathrm{S}(= 1.49598 \times 10^8 \text{ km})$ and the angular velocity is $\omega_\mathrm{S}(= 1.99640 \times 10^{-7} \text{ 1/s})$. The Earth and the Moon rotate along the circular orbits around their barycenter with the angular velocity $\omega_\mathrm{M}(= 2.66498 \times 10^{-6} \text{ 1/s})$ and the distance between the planets is $a_\mathrm{M}(= 3.84400 \times 10^5 \text{ km})$, as shown in Fig.3. The masses of the Sun, Earth and Moon are denoted by $m_\mathrm{S}, m_\mathrm{E}$ and m_M, respectively. The spacecraft and planets move on the same plane.

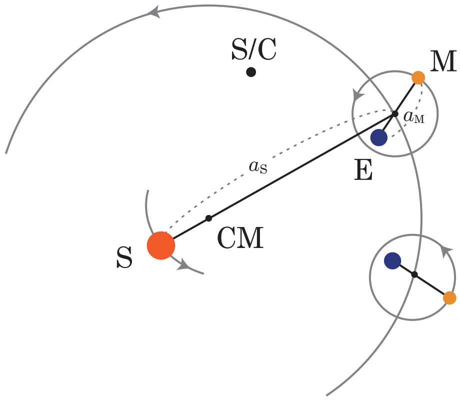

Fig. 3: Bicircular model

Recall that the coupled PRC3BS is an approximated model of the planar bicircular restricted 4-body problem and we consider the coupled system of the Sun-Earth-S/C system with the mass parameter $\mu_\mathrm{S} = m_\mathrm{E}/(m_\mathrm{S}+m_\mathrm{E}) = 3.02319 \times 10^{-6}$ and the Earth-Moon-S/C system with the mass parameter $\mu_\mathrm{M} = m_\mathrm{M}/(m_\mathrm{E}+m_\mathrm{M}) = 1.21536 \times 10^{-2}$. In this paper, the expression for some quantity or set A is given by local coordinates of the Earth-Moon (E-M) rotating frame, while $\bar{A}$ is expressed by local coordinates of the Sun-Earth (S-E) rotating frame. Let $(\bar{x}, \bar{y}, \bar{v}_{\bar{x}}, \bar{v}_{\bar{y}}) \in M$ denote the position and velocity of the spacecraft in the S-E rotating frame.

It follows from (3) that the energy surface in the S-E rotating frame is given by

$$\bar{\mathcal{E}}(\mu_S, \bar{E}_0^{SE}) = \{\bar{w} = (\bar{x}, \bar{y}, \bar{v}_{\bar{x}}, \bar{v}_{\bar{y}}) \in M \mid \bar{E}^{SE}(\bar{w}) = \bar{E}_0^{SE}\},$$

where $\bar{E}_0^{SE}$ denotes a given energy of the Sun-Earth-S/C system. Denoting by $w = (x, y, v_x, v_y) \in M$ the local coordinate in the E-M rotating frame and by also E_0^{EM} a given energy of the Earth-Moon-S/C system, the energy surface is defined as

$$\mathcal{E}(\mu_M, E_0^{EM}) = \{w = (x, y, v_x, v_y) \in M \,|\, E^{EM}(w) = E_0^{EM}\}.$$

We show the transformation between the S-E and E-M rotating frame below. The transformation of time is given by

$$\bar{t} = \frac{\omega_\mathrm{S}}{\omega_\mathrm{M}} t.$$

The transformation of position is

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} 1-\mu_\mathrm{S} \\ 0 \end{pmatrix} + \frac{a_\mathrm{M}}{a_\mathrm{S}} C(t) \begin{pmatrix} x \\ y \end{pmatrix}.$$

In the above,

$$C(t) = \begin{pmatrix} \cos(\theta_\mathrm{M}) & -\sin(\theta_\mathrm{M}) \\ \sin(\theta_\mathrm{M}) & \cos(\theta_\mathrm{M}) \end{pmatrix},$$

where $\theta_\mathrm{M} = (1-\omega_\mathrm{S}/\omega_\mathrm{M})t + \theta_{\mathrm{M}0}$ is the angel between x axis and $\bar{x}$ axis. The coordinate transformation for the velocity phase space, $\tilde{\varphi} : M \times I \to M \times I, (x, y, v_x, v_y, t) \mapsto (\bar{x}, \bar{y}, \bar{v}_{\bar{x}}, \bar{v}_{\bar{y}}, \bar{t})$, is given by

$$\begin{pmatrix} \bar{v}_{\bar{x}} \\ \bar{v}_{\bar{y}} \end{pmatrix} = \frac{a_\mathrm{M}}{a_\mathrm{S}\omega_\mathrm{S}} C(t) \begin{pmatrix} \omega_\mathrm{M} v_x - (\omega_\mathrm{M}-\omega_\mathrm{S})y \\ \omega_\mathrm{M} v_y + (\omega_\mathrm{M}-\omega_\mathrm{S})x \end{pmatrix}.$$

3.2. The Family of Departure Trajectories

For the boundary condition that the spacecraft departs from the LEO (167 km, 7.80713 km/s) by the maneuver ΔV_E, we want to get the departure trajectory in the Sun-Earth-S/C system. Set the departure point on the LEO in the S-E rotating frame as $(\bar{x}, \bar{y}, \bar{v}_{\bar{x}}, \bar{v}_{\bar{y}}) = (1-\mu_\mathrm{S}-\bar{r}_\mathrm{LEO}, 0, 0, -\bar{v}_\mathrm{LEO}) = (0.999953, 0, 0, -0.261364)$, where $\bar{r}_\mathrm{LEO} = 4.37038 \times 10^{-5}$ denotes the altitude of the spacecraft from the Earth's center and $\bar{v}_\mathrm{LEO} = 0.261364$ the velocity of the LEO in the S-E rotating frame. The energy at the point is $\bar{E}_\mathrm{LEO}^{SE} = -1.53501$. It is assumed that ΔV_E is produced in parallel to the velocity of the LEO in Fig.4. Hence the spacecraft is transferred to the patch point $\bar{w}_\mathrm{D} = (\bar{x}_\mathrm{D}, \bar{y}_\mathrm{D}, \bar{v}_{\bar{y}\mathrm{D}}, \bar{v}_{\bar{y}\mathrm{D}}) = (1-\mu_\mathrm{S}-\bar{r}_\mathrm{LEO}, 0, 0, -\bar{v}_\mathrm{LEO}-\Delta V_\mathrm{E}) = (0.999953, 0, 0, -\bar{v}_\mathrm{LEO}-\Delta V_\mathrm{E})$. Note that the energy $\bar{E}^{SE}$ after the maneuver is uniquely obtained by ΔV_E.

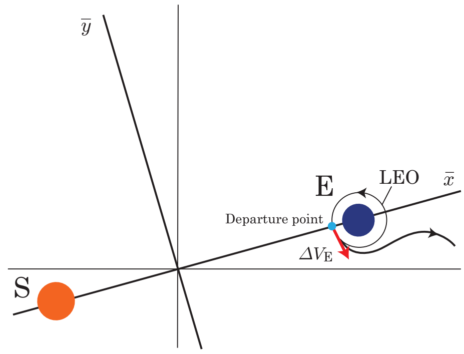

Fig. 4: Configuration of departure point

Here, we construct the non-transit orbit from the patch point as the departure trajectory in order to remain in the Earth region. So, the patch point after the maneuver is required to be outside of the stable manifold $\bar{W}^s_{2,\mathrm{E}}$. Let us find out an energy range (associated with ΔV_E) satisfying this condition. To do this, set the Poincaré section $\bar{U}$ by

$$\bar{U} = \{(\bar{x}, \bar{y}, \bar{v}_{\bar{x}}, \bar{v}_{\bar{y}}) \in \bar{\mathcal{E}}(\mu_\mathrm{S}, \bar{E}_0^{SE}) \mid \bar{x} < 1-\mu_\mathrm{S}, \bar{y} = 0, \bar{v}_{\bar{x}} > 0\}.$$

Set the energy at Lagrangian point $\bar{L}_2$, $(\bar{x}, \bar{y}, \bar{v}_{\bar{x}}, \bar{v}_{\bar{y}}) = (0.999997, 0, 0, 0)$, by $\bar{E}_{\bar{L}_2}^{SE} = -1.50045$, and there exists the neck region within the energy range $\bar{E}_{\bar{L}_2}^{SE} \le \bar{E}_0^{SE}$. Now, let us find out the upper limit of the departure trajectory for this energy region by investigating the stable manifold $\bar{W}^s_{2,\mathrm{E}}$ on $\bar{U}$.

The LEO and the patch point are illustrated together with $\bar{W}^s_{2,\mathrm{E}} \cap \bar{U}$ for the case of $\bar{E}_0^{SE} = -1.50040$ in Fig.5. One can detect that the energy range is $\bar{E}^{SE}_{\bar{L}_2} \leq \bar{E}^{SE}_0 \leq \bar{E}_{D_{\max}} = -1.50040$ where an orbit is to be a non-transit orbit. In particular, the stable manifold $\bar{W}^s_{2,\mathrm{E}}$ with $\bar{E}^{SE}_0 = \bar{E}_{D_{\max}}$ is illustrated in Fig.6. Needless to say, for the region $\bar{E}_{D_{\max}} < \bar{E}^{SE}_0$, the spacecraft is transported to the exterior region since the patch point is inside of $\bar{W}^s_{2,\mathrm{E}}$.

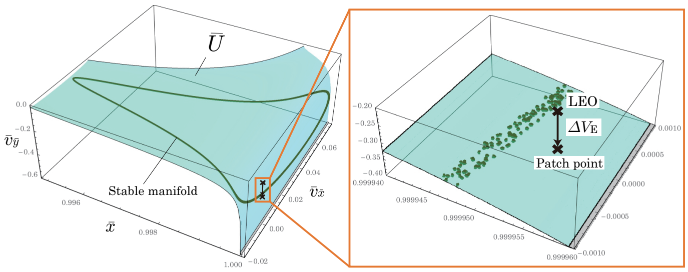

Fig. 5: Boundary of non-transit orbits ($\bar{E}^{SE}_0 = -1.50040$)

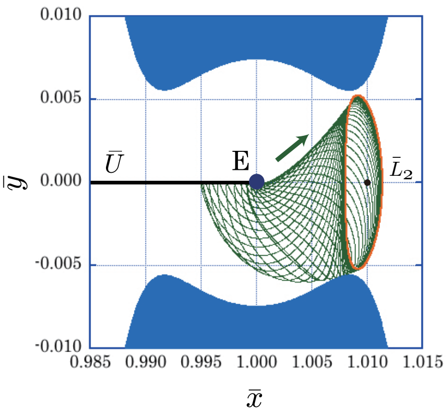

Fig. 6: Stable manifolds $\bar{W}^s_{2,\mathrm{E}}$ ($\bar{E}^{SE}_{\mathrm{D}_{\max}} = -1.50040$)

In this way, we can define the family of the non-transit orbits parametrized by the energy $\bar{E}^{SE}_0$ as

$$\mathcal{D}(\bar{E}^{SE}_0) = \{\phi_{\bar{t}}(\bar{w}_{\mathrm{D}}) \in M \mid \bar{w}_{\mathrm{D}} \in \bar{\mathcal{E}}(\mu_{\mathrm{S}}, \bar{E}^{SE}_0), \\ \bar{E}^{SE}_0 \in [\bar{E}^{SE}_{\bar{L}_2}, \bar{E}^{SE}_{\mathrm{D}_{\max}}]\}.$$

In the above, $\phi_{\bar{t}} : M \to M$ is the flow, where $\bar{t} \in I \subset \mathbb{R}$. Associated with 50 values for the energy $\bar{E}^{SE}_0 \in [\bar{E}^{SE}_{\bar{L}_2}, \bar{E}^{SE}_{\mathrm{D}_{\max}}]$, the family of the non-transit orbits is illustrated in Fig.7.

3.3. The Family of Arrival Trajectories

One needs to consider another boundary condition for the spacecraft; namely, the spacecraft is to be transferred from an arrival trajectory into the LLO (100 km, 1.63346 km/s) by a correction maneuver ΔV_{M} in the Earth-Moon-spacecraft system. Finally, the spacecraft is to arrive at $(x, y, v_x, v_y) = (1-\mu_M - r_{\mathrm{LLO}}, 0, 0, -v_{\mathrm{LLO}}) = (0.983066, 0, 0, -1.58974)$,

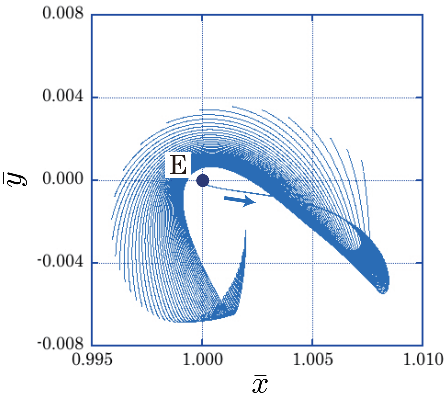

Fig. 7: Family of non-transit orbits from LEO ($\bar{E}^{SE}_0 \in [\bar{E}^{SE}_{\bar{L}_2}, \bar{E}^{SE}_{\mathrm{D}_{\max}}]$)

where the energy at the point is $E_{\mathrm{LLO}} = -2.75466$, $r_{\mathrm{LLO}} = 4.78018 \times 10^{-3}$ indicates the altitude of the spacecraft from the Earth's center and $v_{\mathrm{LLO}} = 1.58974$ the velocity of the spacecraft at the LLO in the E-M rotating frame. We assume the correction maneuver is produced in parallel to the velocity of the LLO in Fig.8. One can determine the patch point as $w_{\mathrm{A}} = (x_{\mathrm{A}}, y_{\mathrm{A}}, v_{x\mathrm{A}}, v_{y\mathrm{A}}) = (1 - \mu_{\mathrm{M}} - r_{\mathrm{LLO}}, 0, 0, -v_{\mathrm{LLO}} - \Delta V_{\mathrm{M}}) = (0.983066, 0, 0, -v_{\mathrm{LLO}} - \Delta V_{\mathrm{M}})$. Since the energy at the LLO is given, i.e., $E_{\mathrm{LLO}} = -2.75466$, one can uniquely determine the energy E^{EM}_0 at the patch point associated with some correction maneuver ΔV_{M}.

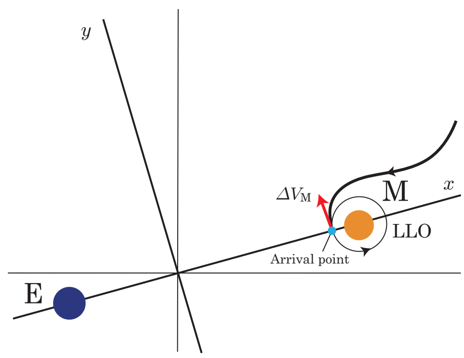

Fig. 8: Configuration of departure point

Here, we construct the transit orbit from the LLO to the exterior region in the backward time as the arrival trajectory. The patch point is so chosen that it is to be inside of the unstable manifold $W^u_{2,\mathrm{M}}$. We shall determine the energy range (associated with ΔV_{M}) to obtain the transit orbit and set the Poincaré section U by

$$U = \{(x, y, v_x, v_y) \in \mathcal{E}(\mu_{\mathrm{M}}, E^{EM}_0) \mid x < 1 - \mu_{\mathrm{M}}, \\ y = 0, v_x < 0\}.$$

Since the energy at the Lagrangian point L_3 is $E_{L_3} = -1.50608$, where L_3 is $(x, y, v_x, v_y) = (-1.00506, 0, 0, 0)$, the upper limit of the energy is given by $E_0^{EM} \leq E_{L_3}$. In this energy range, we illustrate the unstable manifold $W^u_{2,\mathrm{M}}$ on U, namely, the subset $W^u_{2,\mathrm{M}} \cap U$ for the case of $E_0^{EM} = -1.57961$ in Fig.9. So, we can determine the lower limit of the energy as $E^{EM}_{\mathrm{A_{min}}} = -1.57961 \leq E_0^{EM} \leq E^{EM}_{L_3}$. We illustrate $W^u_{2,\mathrm{M}}$ with $E_0^{EM} = E^{EM}_{\mathrm{A_{min}}}$ in Fig.10.

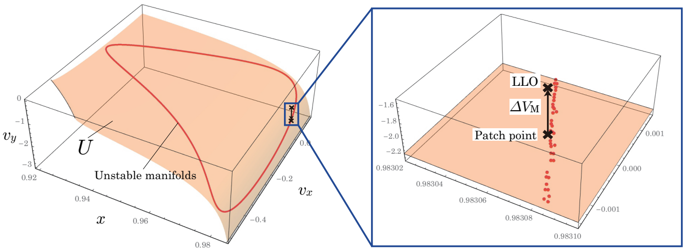

Fig. 9: Boundary of non-transit orbits
($E_0^{EM} = -1.57961$)

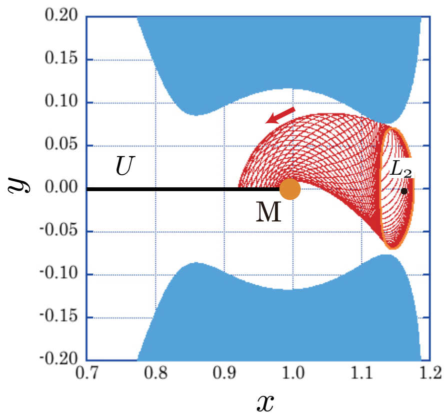

Fig. 10: Stable manifold $W^u_{2,\mathrm{M}}$
($E^{EM}_{\mathrm{A_{min}}} = -1.57961$)

One can obtain the family of the transit orbits parametrized by the energy E_0^{EM}, which is given by

$$\mathcal{A}(E_0^{EM}) = \{\phi_{-t}(w_{\mathrm{A}}) \in M \mid w_{\mathrm{A}} \in \mathcal{E}(\mu_{\mathrm{M}}, E_0^{EM}), \quad E_0^{EM} \in [E^{EM}_{\mathrm{A_{min}}}, E^{EM}_{L_3}]\}.$$

We show the family of the transit orbits associated with 50 values of E_0^{EM} in Fig.11.

3.4. Design of the Transfer Orbit from the LEO to the LLO

In this section, we show how the transfer orbit from the LEO to the LLO can be obtained by patching the departure and arrival trajectories in the previous sections. In particular, we demonstrate that the optimal transfer can be made in the sense that $\Delta V_{\mathrm{P}} = 0$, though the maneuver ΔV_{P} is generally required in the patching.

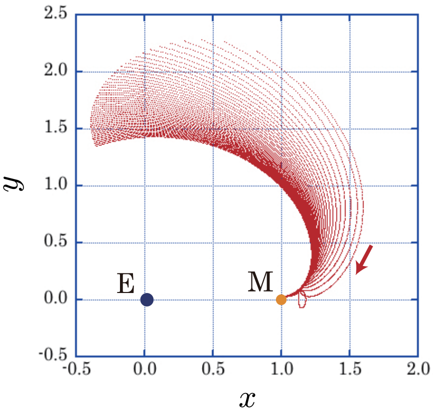

Fig. 11: Family of transit orbits to LLO
($E_0^{EM} \in [E^{EM}_{\mathrm{A_{min}}}, E^{EM}_{L_3}]$)

Let us consider the Sun-Earth-S/C and Earth-Moon-S/C systems in the S-E rotating frame in order to patch the trajectories. The Earth-Moon-S/C system in the S-E rotating frame can be considered to be a non-autonomous system since it depends on the angle $\bar{\theta}_{\mathrm{M}} \in [0, 2\pi)$. We show the arrival trajectories $\tilde{\varphi}(\mathcal{A}(E_0^{EM}))$ at $\bar{\theta}_{\mathrm{M}} = 2.58030$ in addition to the family of the departure trajectories in Fig.12.

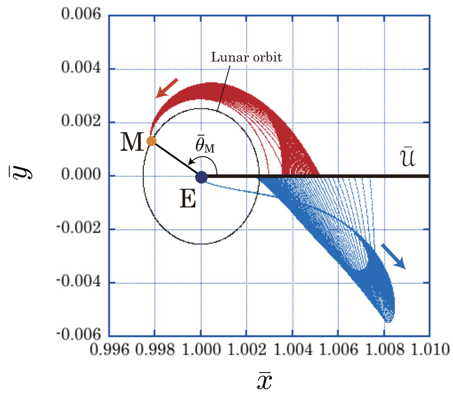

Fig. 12: Departure and arrival trajectories
in the S-E rotating coordinate ($\bar{\theta}_{\mathrm{M}} = 2.58030$)

To determine a patch point, set the Poincaré section on the $\bar{x}$ axis by

$$\bar{\mathbb{U}} = \{(\bar{x}, \bar{y}, \bar{v}_{\bar{x}}, \bar{v}_{\bar{y}}) \mid \bar{x} > 1 - \mu_{\mathrm{S}}, \bar{y} = 0, \bar{v}_{\bar{y}} > 0\}.$$

We show the family of the departure trajectories crossing the section $\bar{\mathbb{U}}$ given by $\mathcal{D}(\bar{E}_0^{SE}) \cap \bar{\mathbb{U}}$ and the subset of the family of the arrival trajectories $\tilde{\varphi}(\mathcal{A}(E_0^{EM})) \cap \bar{\mathbb{U}}$ for $\bar{\theta}_{\mathrm{M}} \in [0, 2\pi)$ in Fig.13. Thus, the patch point can be uniquely chosen at the intersection of $\mathcal{D}(\bar{E}_0^{SE}) \cap \bar{\mathbb{U}}$ and $\tilde{\varphi}(\mathcal{A}(E_0^{EM})) \cap \bar{\mathbb{U}}$ as

$$(\bar{x}_{\mathrm{P}}, \bar{y}_{\mathrm{P}}, \bar{v}_{\bar{x}\mathrm{P}}, \bar{v}_{\bar{y}\mathrm{P}}) = (1.00521, 0, -0.0105875, 0.0173207).$$

The transfer trajectory is obtained by smoothly connecting the departure and arrival trajectories, where the trajectory

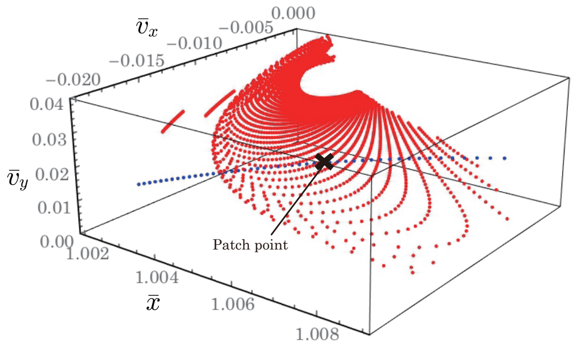

Fig. 13: Patch point and the departure and arrival trajectories ($\bar{\theta}_{\mathrm{M}} \in [0, 2\pi)$) on $\bar{\mathfrak{U}}$

is *optimal* in the sense that the maneuver ΔV_{P} is *zero*. The departure trajectory can be computed from the initial point as

$$(\bar{x}_{\mathrm{D}}, \bar{y}_{\mathrm{D}}, \bar{v}_{\bar{x}\mathrm{D}}, \bar{v}_{\bar{y}\mathrm{D}}) = (0.999953, 0, 0, -0.370838),$$

where the energy is $\bar{E}_0^{SE} = -1.50041$. It follows that $\Delta V_{\mathrm{E}} = 3.270\ \mathrm{km/s}$. The final point of the arrival trajectory in the S-E rotating frame is given by

$$(\bar{x}_{\mathrm{A}}, \bar{y}_{\mathrm{A}}, \bar{v}_{\bar{x}\mathrm{A}}, \bar{v}_{\bar{y}\mathrm{A}}) = (0.997859, 0.00134456, 0.0238594, 0.0379475),$$

and the angle at the final point is $\bar{\theta}_{\mathrm{M}} = 2.58030$. The energy of the arrival trajectory is $E_0^{EM} = -1.56243$ and hence the maneuver is $\Delta V_{\mathrm{M}} = 0.642\ \mathrm{km/s}$.

As in Fig.14, it is clear that the patch point is outside of the unstable manifold $\bar{W}^u_{2,\mathrm{E}}$ of the Sun-Earth-S/C system and inside of the stable manifold $W^s_{2,\mathrm{M}}$ of the Earth-Moon-S/C system on $\bar{\mathfrak{U}}$.

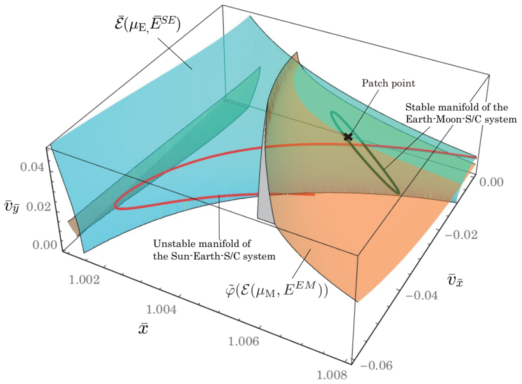

Fig. 14: Patch point and invariant manifolds on $\bar{\mathfrak{U}}$

We illustrate the obtained transfer from the LEO to the LLO in the S-E rotating frame in Fig.15 and also that in the E-M rotating frame in Fig.16. In this transfer, the flight time $T = 102$ d is required.

Finally, we compare the proposed transfer with the Hohmann transfer and a transfer obtained by the previous coupled PRC3BS [2] in Table 1. We obtain the Hohmann transfer as the elliptic orbit connecting with the LEO and the lunar orbit. Using the previous approach of the coupled PRC3BS, the energies of the Sun-Earth-S/C and E-M-S/C systems are $\bar{E}_0^{SE} = -1.50037$ and $E_0^{EM} = -1.58340$ respectively. As in Table 1, it follows that the total maneuver of the proposed approach is $0.067\ \mathrm{km/s}$ fewer than the Hohmann transfer. In addition, the proposed transfer does not require any maneuver in patching comparing with the previous coupled PRC3BS.

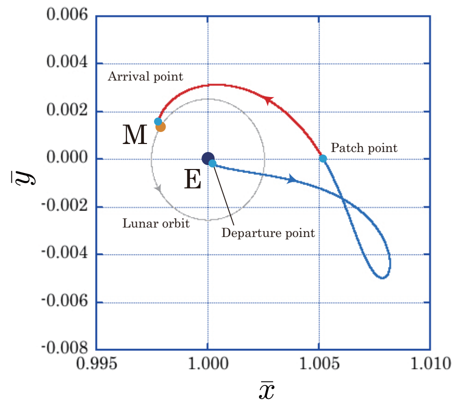

Fig. 15: Transfer from the LEO to the LLO in the SE rotating coordinate

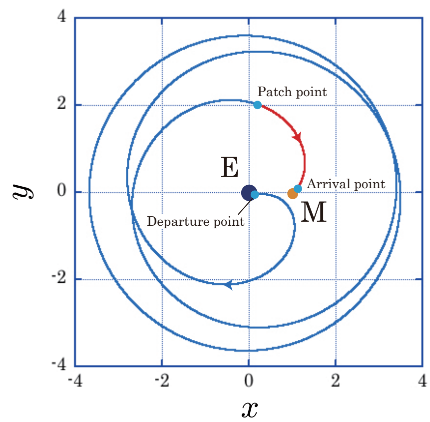

Fig. 16: Transfer from the LEO to the LLO in the EM rotating coordinate

Table 1: Maneuver (ΔV [km/s]) and flight time (T [d])

Transfer	ΔV_{E}	ΔV_{M}	ΔV_{P}	T
Hohmann	3.141	0.838	–	5
Coupled PRC3BS[2]	3.537	1.989	0.098	179
Proposed approach	3.270	0.642	0	102

4. CONCLUSION

In this paper, we have shown design of a low energy transfer under some given boundary conditions where the spacecraft departs from a low Earth orbit (LEO) and arrives at a low lunar orbit (LLO). The transfer has been developed by using the coupled planar restricted circular 3-body system, namely, the Sun-Earth-spacecraft and Earth-Moon-spacecraft systems. First, we have constructed the family of the departure trajectories so that the trajectories are to be the non-transit orbits from the LEO existed outside of the stable manifold in the Sun-Earth-spacecraft system. The family of the arrival trajectories has been obtained by the transit orbits inside of the unstable manifold in the Earth-Moon-spacecraft system. By parametrizing the family of the arrival trajectories by the angle $\bar{\theta}_{\mathrm{M}}$ in the Sun-Earth rotating frame, we have obtained the optimized patch point at which the departure and arrival trajectories are connected smoothly with $\Delta V_{\mathrm{P}} = 0$. Finally, we have illustrated the low energy transfer from the LEO to the LLO and the validity of the proposed approach in comparison with other approaches.

5. ACKNOWLEDGMENT

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