– ICATT 2016 - Student Session –

Dynamics in the center manifold around equilibrium points in Periodically Perturbed Three-Body Problems

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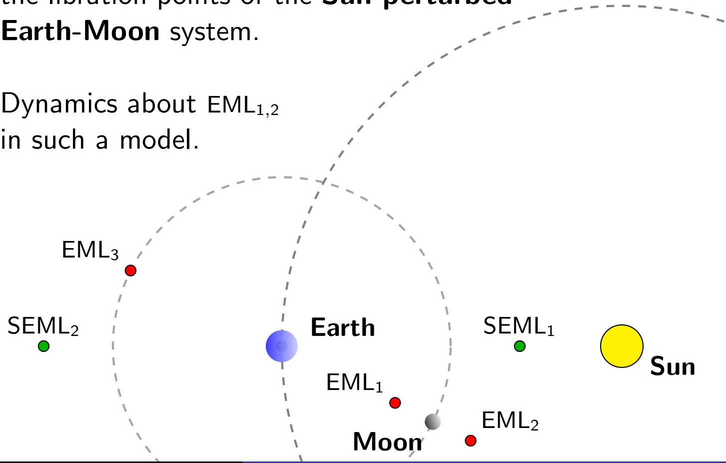
March 15^{th}, 2016

Overview

Framework and objectives

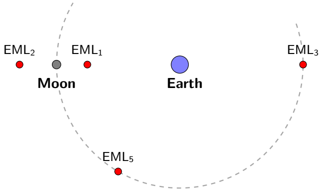
Long term: Near-systematic tool for the motion about & between the libration points of the **Sun-perturbed Earth-Moon** system.

Short term: Dynamics about $EML_{1,2}$ in such a model.

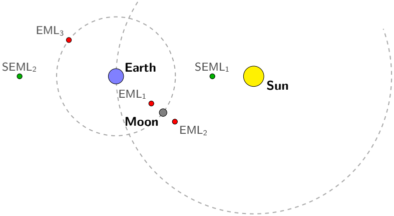


A hierarchy of models

(A) Circular Restricted Three-Body Problem (RTBP)

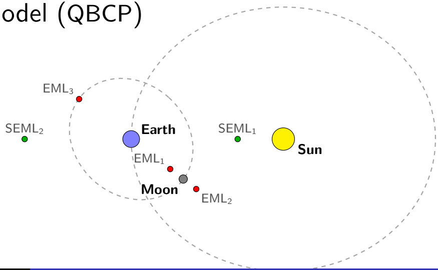


(B) Bicircular Four-Body Model

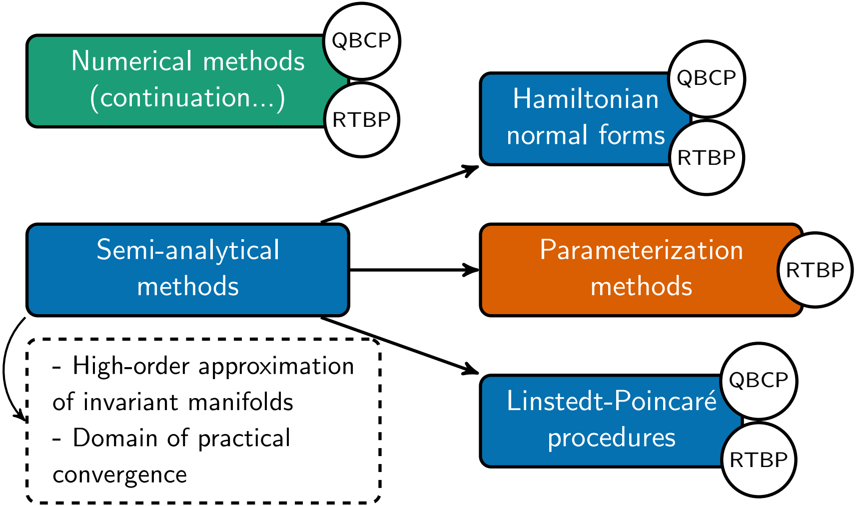


(C) Quasi-Bicircular Four-Body Model (QBCP)

- ✓ Coherent
- ✓ Formally equivalent to (B)



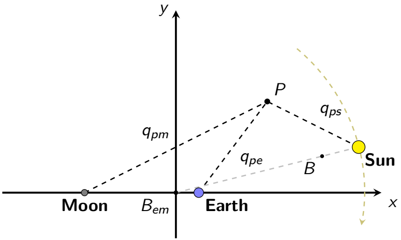
A variety of methods in the literature



The Quasi-Bicircular Problem (QBCP)

Periodic Hamiltonian in the Earth-Moon synodical frame

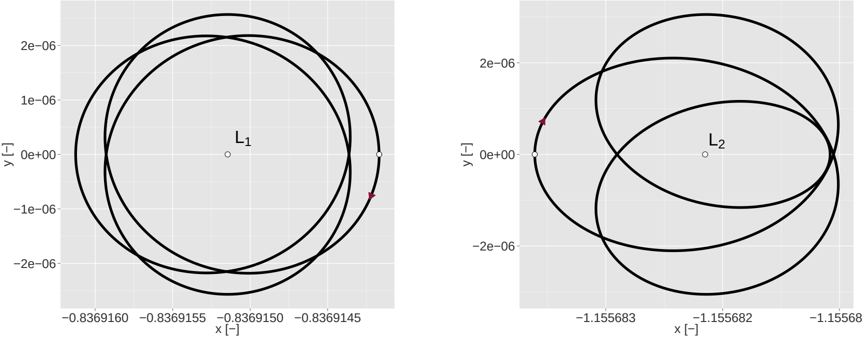
$$H(\mathbf{z}, \theta) = \frac{1}{2}\alpha_1(p_x^2 + p_y^2 + p_z^2) + \alpha_2(p_x x + p_y y + p_z z) + \alpha_3(p_x y - p_y x) + \alpha_4 x + \alpha_5 y - \alpha_6 \left(\frac{1-\mu}{q_{pe}} + \frac{\mu}{q_{pm}} + \frac{m_s}{q_{ps}} \right)$$



- μ the Earth-Moon mass ratio
- m_s the mass of the Sun
- α_k trigonometric functions in the variable $\theta = \omega_s t$
- ω_s the pulsation of the Sun.

Earth-Moon $L_{1,2}$ in the QBCP

2π-periodic dynamical equivalents of the Earth-Moon libration points $L_{1,2}$, in Earth-Moon coordinates.



Overview

First step: getting an **autonomous diagonal** order one

- $\exists$ 2π-periodic symplectic change of coordinates of the form:

$$\mathbf{z} = P(\theta)\hat{\mathbf{z}} + V(\theta)$$

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In the new coordinates $\hat{\mathbf{z}} = \begin{pmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 & \hat{y}_1 & \hat{y}_2 & \hat{y}_3 \end{pmatrix}^T$

- Hamiltonian:

$$\hat{H}(\hat{\mathbf{z}}, \theta) = \omega_1 \mathbf{i} \hat{x}_1 \hat{y}_1 + \omega_2 \hat{x}_2 \hat{y}_2 + \omega_3 \mathbf{i} \hat{x}_3 \hat{y}_3 + \sum_{k \geq 3} \hat{H}_k(\hat{\mathbf{z}}, \theta)$$

- The origin becomes a fixed point.

First step: getting an **autonomous diagonal** order one

- $\exists$ 2π-periodic symplectic change of coordinates of the form:

$$\mathbf{z} = P(\theta)\hat{\mathbf{z}} + V(\theta)$$

Precision bottleneck

In the new coordinates $\hat{\mathbf{z}} = \begin{pmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 & \hat{y}_1 & \hat{y}_2 & \hat{y}_3 \end{pmatrix}^T$

- Hamiltonian:

$$\hat{H}(\hat{\mathbf{z}}, \theta) = \omega_1 \mathbf{i} \hat{x}_1 \hat{y}_1 + \omega_2 \hat{x}_2 \hat{y}_2 + \omega_3 \mathbf{i} \hat{x}_3 \hat{y}_3 + \sum_{k \geq 3} \hat{H}_k(\hat{\mathbf{z}}, \theta)$$

- The origin becomes a fixed point.

Parameterization of the center manifold (1)

- Linearized vector field: $D\hat{F}(0) = \text{diag}(\mathbf{i}\omega_1, \omega_2, \mathbf{i}\omega_3, -\mathbf{i}\omega_1, -\omega_2, -\mathbf{i}\omega_3)$.
- Isolating the center part:

$$L = \begin{pmatrix} \mathbf{i}\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathbf{i}\omega_3 & 0 & 0 \\ 0 & 0 & -\mathbf{i}\omega_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{i}\omega_3 \end{pmatrix}$$

L spans the 4-dimensional subspace $V^L \subset \mathbb{C}^6$ tangent to the center manifold $\mathcal{W}_c$ about the origin.

- **Goal**: compute

$$\begin{array}{cccccc} \hat{W} & : & \mathbb{C}^4 \times \mathbb{R} & \rightarrow & \mathbb{C}^6 \\ & & (\mathbf{s}, \theta) & \mapsto & \hat{W}(\mathbf{s}, \theta) \end{array}$$

High-order parameterization of $\mathcal{W}_c$, starting with $\hat{W}_0(\mathbf{s}, \theta) = 0$, $\hat{W}_1(\mathbf{s}, \theta) = L\mathbf{s}$.

Parameterization of the center manifold (2)

- Parameterization form: Fourier-Taylor (FT) series.

$$\hat{W}^1(\mathbf{s},\theta) = \sum_{k\geq 1} \hat{W}^1_k(\mathbf{s},\theta) = \sum_{k\geq 1} \sum_{r_1+\cdots+r_4=k} \underbrace{w^1_r(\theta)}_{\text{Fourier series}} \quad s_1^{r_1}\dots s_4^{r_4}$$

- Dynamics on the manifold: $\dot{\mathbf{s}} = f(\mathbf{s},\theta)$, $f(0) = 0$.

Parameterization method: an iterative procedure

- $(\hat{W}, f)$ satisfy the invariance equation:

$$\hat{F}\left(\hat{W}(\mathbf{s},\theta),\theta\right) = D_{\mathbf{s}}\hat{W}\left(\mathbf{s},\theta\right) f\left(\mathbf{s},\theta\right) + \frac{\partial \hat{W}}{\partial t}\left(\mathbf{s},\theta\right) \qquad (1)$$

- At order k: substitute $(\hat{W}_{k-1}, f_{k-1})$ in (1) and find the k-homogeneous terms that solve $(1)_k$.

Solving the invariance equation

Computing $(\hat{W}_k, f_k)$ in different *styles* (Haro, 2008):

Graph style: $\hat{W}_k$ as simple as possible.

- ✓ Limit the number of small divisors.
- ✓ Easy projection $\mathbf{s} = \hat{W}_k^{-1}(\hat{\mathbf{z}})$.
- ✗ $f(\mathbf{s}, \theta)$ is a full Fourier-Taylor series.

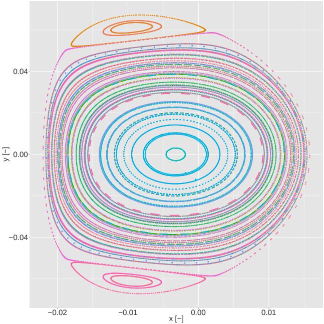
Normal form style: f_k as simple as possible.

- ✓ Possible autonomous f up to a medium order.
- ✗ Numerous small divisors ⇒ divergence rate increased.

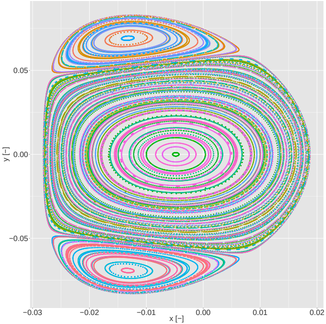
Overview

Poincaré maps: basic principle (autonomous case)

- Intersection with a lower-dimensional subspace: $z = 0$, $p_z > 0$.
- In practice: $\mathbf{z}(t)$ is regularly projected on the center manifold.
- In the autonomous case: energy slices. $\delta H_0 = H(\mathbf{z}) - H(L_1) = cst$



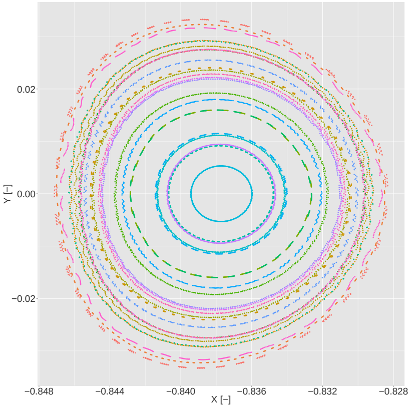
CRTBP, EML_1, $\delta H_0 = 0.01$



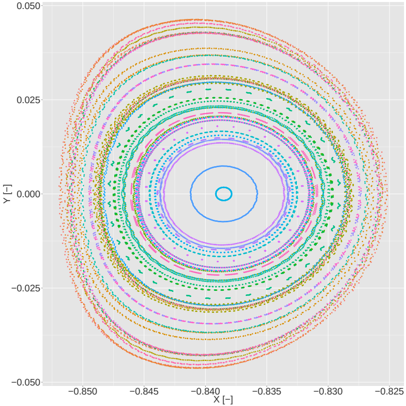
CRTBP, EML_1, $\delta H_0 = 0.015$

QBCP EML_1 case

- Graph style is used.
- Energy no longer constant but bounded.

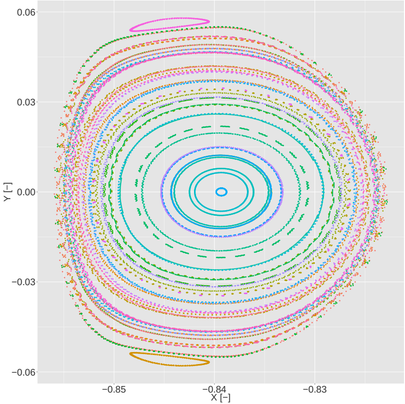


$\delta H_0 = H(\mathbf{z}, 0) - H(L_1, 0) = 0.0025$

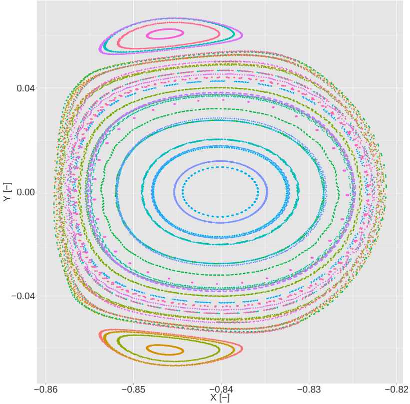


$\delta H_0 = 0.005$

QBCP EML_1 case



$\delta H_0 = 0.0075$



$\delta H_0 = 0.01$

QBCP EML_2 case

- Low energy: easily obtained up to $\delta H_0 \sim 0.008$.
- Higher energy: apparent precision decay. Who is to blame? more
  - Low-order resonances. more
  - Inherent properties.

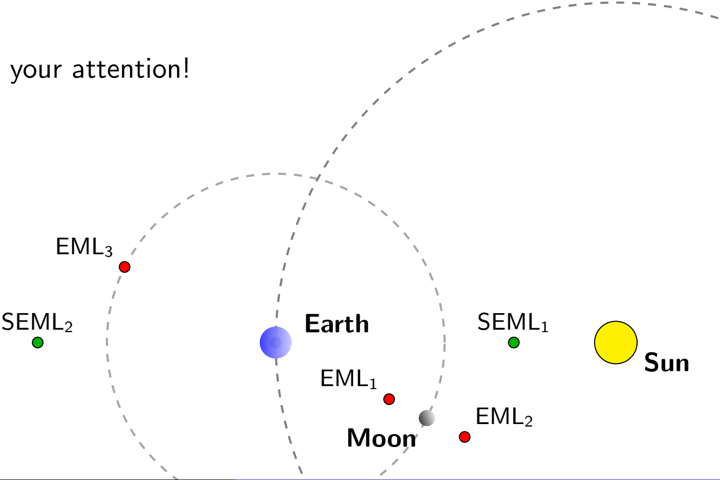
Conclusion

- Example of parameterization of invariant manifolds in the Sun-Earth-Moon system.
  - Flexibility of the method (*styles*).
  - Work in lower dimension (4) than usual normal form procedures (6).
  - Numerically challenging in the non-autonomous case.
- Compact tool for the description of the neighborhood of EML_2 and extended neighborhood of EML_1.
- Works very well in the SEML case (smaller perturbation).

Questions

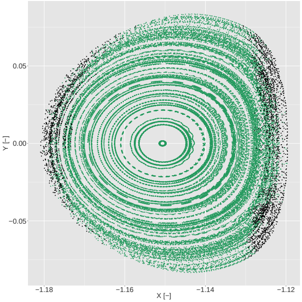
Thank you for your attention!

Questions?

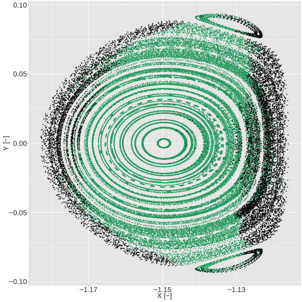


QBCP EML$_2$ case

- Low energy: easily obtained up to $\delta H_0 \sim 0.008$.
- Higher energy: apparent precision decay $(e_P(t) = |\mathbf{z}(t) - \mathbf{z}_p(t)|)$.

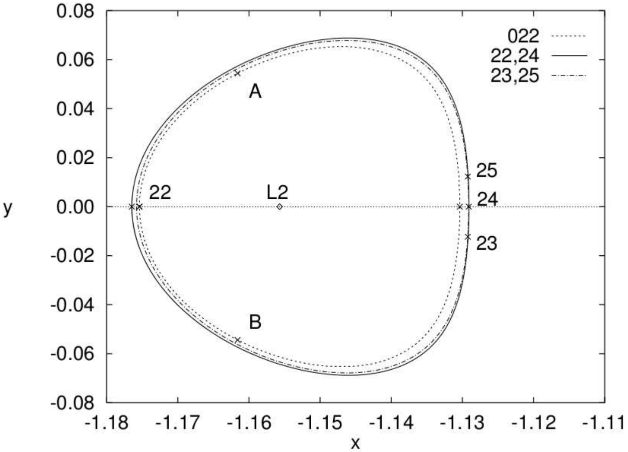


$e_P(t) < 10^{-6}$, $\delta H_0 = 0.01$



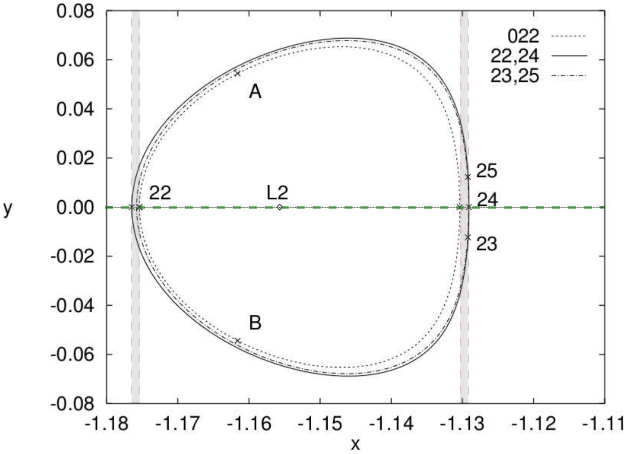
$e_P(t) < 10^{-6}$, $\delta H_0 = 0.012$

Resonances about EML$_2$ in the QBCP



$2\omega_s$-resonant orbits around EML$_2$. From Andreu, 1998.

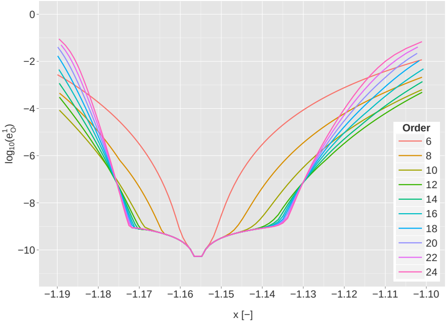
Resonances about EML$_2$ in the QBCP



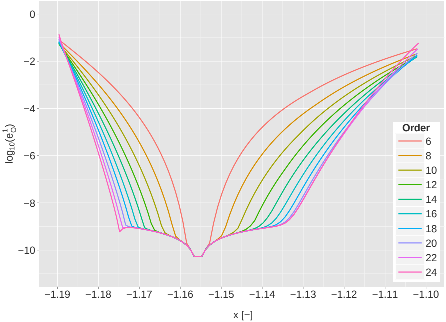
$2\omega_s$-resonant orbits around EML$_2$. From Andreu, 1998.

Accuracy about EML_2 in the QBCP

- Orbital error: $e_O(t, \mathbf{s}_0) = |W(\mathbf{s}(t)) - \mathbf{z}(t)|_\infty$.



Normal form style



Graph style

- There is no miracle!