Modeling and Performance Evaluation of Multistage Launch Vehicles through Firework Algorithm

M. Pallone, M. Pontani, P. Teofilatto

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Introduction (1)

This work purpose is to investigate a simple open-loop guidance strategy applied to a multistage launch vehicle (Scout rocket) to achieve:

- yield a reasonable solution for the ascent trajectory
- fast computation time
- simplification of the launch system as much as possible, including the guidance algorithms
- simple adaptation to similar problem
- good first guess solution to more sophisticated algorithms

The optimization method used in this work is the *firework algorithm*, a novel swarm intelligence heuristic algorithm based on the explosion of the fireworks in a night sky.

Introduction (2)

The technique described in this work is applied to a four-stage rocket, whose two-dimensional trajectory is composed of the following thrust phases and coast arcs:

1. first stage propulsion
2. second stage propulsion
3. third stage propulsion
4. coast arc (after the third stage separation)
5. fourth stage thrust phase

- To simplify the open loop guidance law employed for the first three stages, the aerodynamic angle of attack is assumed constant for each stage
- In the last stage thrust phase the problem of minimizing the propellant is solved defining an Hamiltonian function which is minimized through the Pontryagin minimum principle

Rocket Modeling: Scout Launcher Vehicle

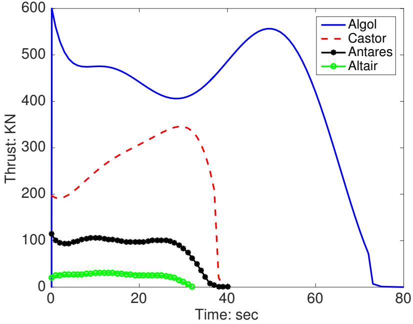
The mass distribution of the launcher vehicle can be described in terms of masses of subrockets i.

- $m_0^{(i)}$, initial mass
- $m_S^{(i)}$, structural mass
- $m_P^{(i)}$, propellant mass
- $m_U^{(i)}$, payload mass

For the last stage, minimizing the propellant mass is equal to maximizing the payload mass. For each subrocket: $m_U^{(i)} = m_0^{(i+1)}$.

i	$m_0^{(i)}$	$m_S^{(i)}$	$m_P^{(i)}$	$m_U^{(i)}$
1	21643 kg	1736 kg	12810 kg	6897 kg
2	6897 kg	915 kg	3749 kg	2033 kg
3	2033 kg	346 kg	1173 kg	514 kg

Propulsive Thrust



The propulsive characteristic of the launch vehicle:

- thrust magnitude $T^{(j)}$, interpolating experimental data at discrete time
- specific impulse $I_{SP}^{(j)}$

the superscript j referring to the stage number

j	1	2	3	4
$Isp^{(j)}$	260 s	288 s	284 s	270 s

Aerodynamics

Aerodynamic modeling is composed of two steps:

1. derivation of C_D and C_L at a relevant number of Mach number and angles of attack (DATCOM)
2. fourth degree polynomial interpolation

Regarding the *DATCOM* database (1), the grid used for *each* subrocket is:

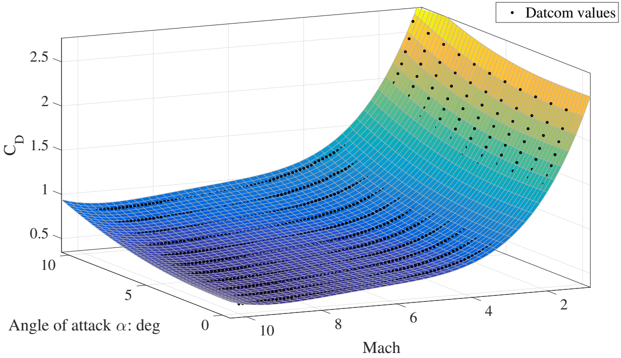
- Subsonic case: $M \in [0, 0.8]$ and $\alpha \in [0, 10]$
- Transonic case: $M \in [0.8, 1.2]$ and $\alpha \in [0, 10]$
- Supersonic case: $M \in [1.2, 10]$ and $\alpha \in [0, 10]$

The aerodynamic forces are the lift and drag forces:

$$L = \frac{1}{2} C_L(\alpha, M) S^{(i)} \rho v^2$$

$$D = \frac{1}{2} C_D(\alpha, M) S^{(i)} \rho v^2$$

Aerodynamics: Fourth Degree Polynomial Interpolation (1)



$$
\begin{aligned}
C_k = {} & C_{00,k} + C_{10,k}\alpha + C_{01,k}M + C_{20,k}\alpha^2 + + C_{11,k}\alpha M + C_{02,k}M^2 \\
& + C_{30,k}\alpha^3 + C_{21,k}\alpha^2 M + C_{12,k}\alpha M^2 + C_{03,k}M^3 + C_{40,k}\alpha^4 \\
& + C_{31,k}\alpha^3 M + C_{22,k}\alpha^2 M^2 + C_{13,k}\alpha M^3 + C_{04,k}M^4
\end{aligned}
$$

with k=L,D.

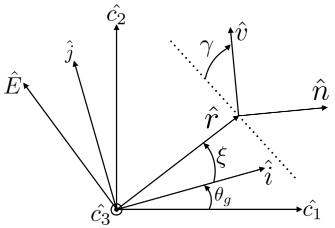
Aerodynamics: Fourth Degree Polynomial Interpolation (2)

	subsonic						supersonic					
	subrocket 1		subrocket 2		subrocket 3		subrocket 1		subrocket 2		subrocket 3	
	C_D	C_L	C_D	C_L	C_D	C_L	C_D	C_L	C_D	C_L	C_D	C_L
C_{00}	0.213	0.141	0.160	0.004	0.113	0.002	6.024	0.098	4.284	-0.515	4.814	-0.381
C_{10}	-0.306	5.638	-0.008	1.714	0.015	1.826	0.810	22.290	1.382	9.774	0.862	4.495
C_{01}	0.017	-1.857	0.961	-0.041	1.055	-0.020	-4.449	-0.202	-2.479	0.608	-2.855	0.466
C_{20}	7.398	6.636	1.959	9.338	1.908	3.333	-5.188	-98.280	-9.404	-86.950	-5.682	-39.830
C_{11}	2.781	10.440	0.092	0.294	-0.114	-0.237	-0.250	-7.475	-0.942	-4.251	-0.583	-1.072
C_{02}	-0.073	6.886	-3.484	0.120	-3.828	0.061	1.481	0.114	0.815	-0.260	0.974	-0.203
C_{30}	5.448	-6.733	7.732	-2.092	2.473	2.403	56.290	422.40	55.090	443.40	31.140	201.0
C_{21}	-4.116	-6.103	-0.330	-3.803	0.370	-2.110	5.577	41.750	3.890	34.230	2.599	15.590
C_{12}	-6.230	-26.20	-0.074	0.373	0.327	1.569	-0.031	1.059	0.201	0.705	0.132	0.044
C_{03}	-0.361	-9.520	3.804	-0.132	4.25	-0.070	-0.227	-0.024	-0.120	0.048	-0.155	0.037
C_{40}	-9.063	-32.270	-2.443	-7.244	-1.494	-15.90	-50.640	-656.60	-49.710	-765.50	-36.330	-353.70
C_{31}	-1.953	14.090	3.628	6.947	2.080	2.065	-3.315	-51.920	-3.605	-47.130	-1.794	-20.810
C_{22}	5.965	2.232	-0.517	5.463	-1.142	3.364	-0.608	-3.569	-0.323	-2.867	-0.263	-1.344
C_{13}	3.866	19.890	-0.008	-1.335	-0.229	-2.122	0.009	-0.043	-0.014	-0.029	-0.010	0.012
C_{04}	0.769	4.373	-0.492	0.048	-0.638	0.027	0.013	0.002	0.007	-0.003	0.009	-0.002

Rocket Dynamics (1)

Simplifying assumptions:

- equatorial trajectory
- launch eastward
- launch vehicle as a point of mass
- two-degree-of-freedom problem



Under these assumption the following frames are defined:

- *ECI*: Earth centered inertial frame $[\hat{c}_1, \hat{c}_2, \hat{c}_3]$
- *ECEF*: Earth centered Earth fixed frame $[\hat{i}, \hat{j}, \hat{k}(= \hat{c}_3)]$, rotates with the angular speed $\omega_E \hat{k}$ with respect to *ECI* frame
- *NVH*: Relative motion frame $[\hat{n}, \hat{v}, \hat{h}]$

with the flight path angle γ, the geographical longitude ξ and the angular position between ECI-ECEF frames is $\theta_g(t) = \theta_g(\bar{t}) + \omega_E(t - \bar{t})$

Rocket Dynamics (2)

- the relation between the velocity in rotating frame and inertial frame is:

$$\mathbf{v}_I = \mathbf{v} + \omega_E \times \mathbf{r}$$

- as the trajectory lies in the equatorial plane only γ suffices to identify the velocity direction

$$\mathbf{v} = v[\sin\gamma, \cos\gamma][\hat{r}, \hat{E}]^T$$

$$\mathbf{r} = r[\cos\xi, \sin\xi][\hat{r}, \hat{E}]^T$$

- the overall aerodynamic force is conveniently written in $[\hat{n}, \hat{v}, \hat{h}]$

$$\mathbf{A} = L\hat{n} - D\hat{v}$$

Equation of Motion - Subrocket 1-2-3

The two dimensional equations of dynamics (in rotating coordinates) are written in terms of its radius r, flight-path angle γ, velocity v and mass m $\{\mathbf{x}_R\ ([r\ \gamma\ v\ m]^T)\}$. For each subrocket:

$$\begin{cases} \dot{r} = v \sin\gamma \\ \dot{\gamma} = \frac{T}{mv}\sin\alpha_T + \left(\frac{v}{r} - \frac{\mu_E}{r^2 v}\right)\cos\gamma + \frac{L}{mv} + 2\omega_E + \frac{\omega_E^2 r}{v}\cos\gamma \\ \dot{v} = \frac{T}{mv}\cos\alpha_T - \frac{\mu_E}{r^2}\sin\gamma - \frac{D}{m} + \omega_E^2 r\sin\gamma \\ \dot{m} = -\frac{T}{I_{sp} g_0} \end{cases}$$

- α_T refers to the thrust angle
- μ_E ($=398600.4\ km^3/sec^2$) is the Earth gravitational parameter

The thrust vector is coplanar with $\mathbf{r}$ and $\mathbf{v}$ so α_T suffices to define its direction, which is taken clockwise from $\mathbf{v}$. The initial condition are:

$$\begin{cases} r(0) = R_E \quad \gamma(0) = 86\ deg \\ v(0) = 0.001\ km/sec \quad m(0) = m_0^{(1)} \end{cases} \tag{1}$$

Equation of Motion - Orbital elements (1)

The orbital elements a and e do not vary during the coast arc. Hence, they can be computed at separation of the third stage through the following steps:

1. derivation of the inertial state variables (r_I, γ_I, v_I) from the relative state variables (r, γ, v)
2. derivation of the orbital elements (a, e, f) from the inertial state variables (r_I, γ_I, v_I)

Equation of Motion - Orbital elements (2)

The velocity vectors $\mathbf{v}_I$ and $\mathbf{v}$ have the following expression in the rotating frame $(\hat{r}, \hat{E}, \hat{N})$:

$$\mathbf{v}_I = [\sin\gamma_I, \cos\gamma_I][\hat{r}, \hat{E}]^T \tag{2}$$

and

$$\mathbf{v} = [\sin\gamma, \cos\gamma][\hat{r}, \hat{E}]^T \tag{3}$$

Due to this and the fact that $\omega_E \times \mathbf{r} = \omega_E r \hat{E}$, Equation (**??**) yields to two simple relations:

$$v_I = \sqrt{v^2 + (\omega_E r)^2 + 2v\omega_E r \cos\gamma} \tag{4}$$

$$\gamma_I = \arcsin \frac{v \sin\gamma}{v_I} \tag{5}$$

Equation of Motion - Coast Arc

the true anomaly variation Δf suffices to describe the rocket dynamics. In fact, if t_{CO} represents the ignition time of the fourth stage, then $f_4 \equiv f(t_{CO}) = f(t_{b3}) + \Delta f$. The orbital elements a and e do not vary during the coast arc.

Equation of Motion - Subrocket 4

During the propulsion phase, the fourth stage motion can be described through the use of the following equations that regard r, v_I and γ_I in the inertial frame

$$\begin{cases} \dot{r}_I = v_I \sin\gamma_I \\ \dot{\gamma}_I = \frac{T}{m}\frac{\sin\alpha_T}{v} + \frac{\mu_E}{r_I^2 v_I}\cos\gamma_I + \frac{L}{m v_I} \\ \dot{v}_I = \frac{T}{m}\frac{\cos\alpha_T}{v_I} - \frac{\mu_E}{r_I^2}\sin\gamma_I - \frac{D}{m} \\ \dot{m} = -\frac{T}{I_{sp} g_0} \end{cases} \tag{6}$$

With the initial conditions $[r(t_{CO}), v_I(t_{CO}), \gamma_I(t_{CO})]$ and α_T the optimal thrust pointing angle.

Formulation of the Problem

Method of Solution - Hamiltonian

$$
\begin{aligned}
H \equiv \lambda_1 x_2 \sin x_3 + \lambda_2 \left[\frac{T}{m}\frac{\cos\alpha_T}{x_2} - \frac{\mu_E}{x_1^2}\sin x_3\right] \\
\lambda_3 \left[\frac{T}{m}\frac{\sin\alpha_T}{x_2} + \left(\frac{x_2}{x_1} - \frac{\mu_E}{x_1^2 x_2}\right)\cos x_3\right]
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\Phi \equiv (t_f - t_{CO}) + \nu_1 \left[x_{10} - \frac{a_3(1-e_3^2)}{1+e_3\cos f_4}\right] \\
+ \nu_2 \left[x_{20} - \sqrt{\frac{\mu_E}{a_2(1-e_3^2)}}\sqrt{1+e_3^2+2e_3\cos f_4}\right] \\
+ \nu_3 \left[x_{30} - \arctan\frac{e_3\sin f_4}{1+e_3\cos f_4}\right] + \nu_4\left[x_{1f} - R_d\right] \\
+ \nu_5 \left[x_{2f} - \sqrt{\frac{\mu_E}{R_d}}\right] + \nu_6 x_{3f}
\end{aligned}
\tag{8}
$$

Method of solution - Algorithm

1. given $[r(0), v(0), \gamma(0), m(0)]$ and $\{\alpha_1, \alpha_2, \alpha_3\}$, the state equations are integrated numerically for each subrocket until the third stage burnout time
2. the coast arc is computed analytically using Equations (**??**)-(**??**)
3. for the upper stage the control variable is expressed as a function of the costate through the Equations (**??**)-(**??**)
4. the value of λ_{30} is calculated by means of Equation (**??**), after picking the unknown values of the remaining Lagrange multipliers at the initial time (λ_{10} and λ_{20}), and the true anomaly f_4
5. Equations (**??**)-(**??**) are used together with Equations (**??**)-(**??**). The respective initial conditions are known once the parameters f_4 (for the state equations) and $\{\lambda_{10}, \lambda_{20}, \lambda_{30}\}$ (for the adjoint equations) are specified
6. the inequality condition in Equation (**??**) (not expanded for the sake of brevity) and the conditions at injection (Equation (**??**)) are evaluated.

Numerical Results - Optimal Set

The boundaries for the optimization set are:

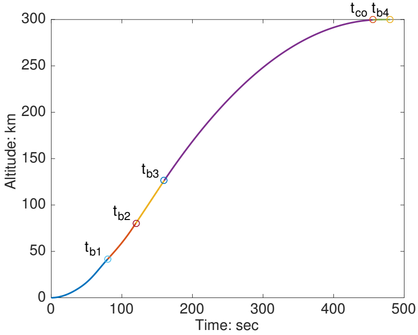
- $0 \leq \alpha_i \leq 10$ deg (i=1,2,3)
- $-1 \leq \lambda_{k0} \leq 1$ (k=1,2,3)
- $0 \leq \Delta f \leq \pi$
- $1/\mathrm{TU} \leq t_f \leq 30/\mathrm{TU}$

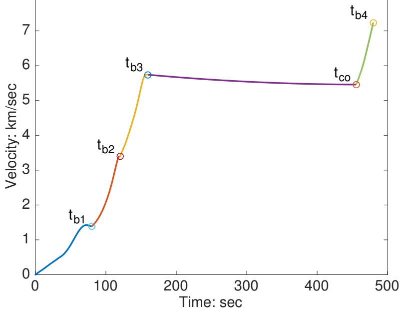
The case test that has been considered has $R_d = R_E + 300$ km. The main optimization results are:

α_1	3.65 deg
α_2	8.86 deg
α_3	9.85 deg
λ_{01}	-0.17958
λ_{02}	-0.41587
Δf	15.33 deg
t_f	24.44 sec

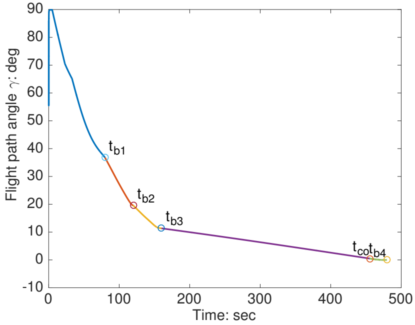
- the coast arc duration is Δt_{CO}=295.21 sec
- the final payload mass is 223.87 kg, including 38.33 kg of structural mass

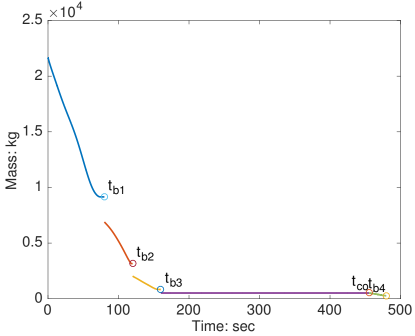
Numerical Results - Altitude and Velocity



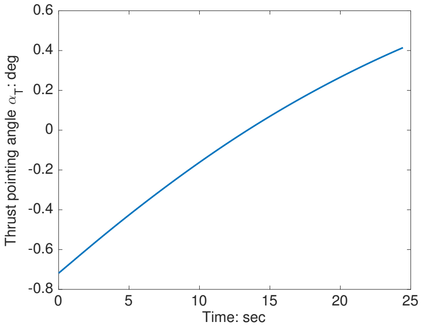


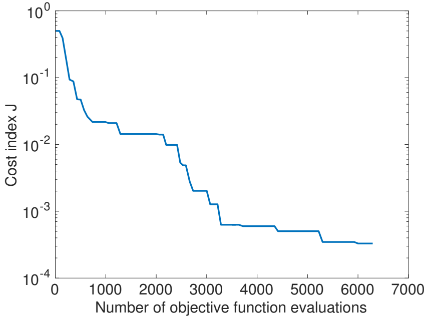
Numerical Results - Flight Path Angle and Mass





Numerical Results - Thrust Pointing Angle and Cost Function Index





Conclusive Remarks

- This work proposes and successfully applies a simple technique for generating near-optimal two-dimensional ascending trajectories for multistage rockets, for the purpose of performance evaluation
- only existing routines and a simple implementation of firework algorithm are employed, in conjunction with the analytical necessary conditions for optimality, applied to the upper stage trajectory
- With regard to the problem at hand, the unknown parameters are:
  1. the aerodynamic angles of attack of the first three stages
  2. the coast time interval
  3. the initial values of the adjoint variables conjugate to the upper stage dynamics
  4. the thrust duration of the upper stage

The numerical results prove that this methodology is rather robust, effective, and accurate, and allows evaluating the performance attainable from multistage launch vehicles with accurate aerodynamic and propulsive modeling and appear suitable as guess for more refined algorithms.