The true nature of the equilibrium for geostationary objects, applications to the high area-to-mass ratio debris

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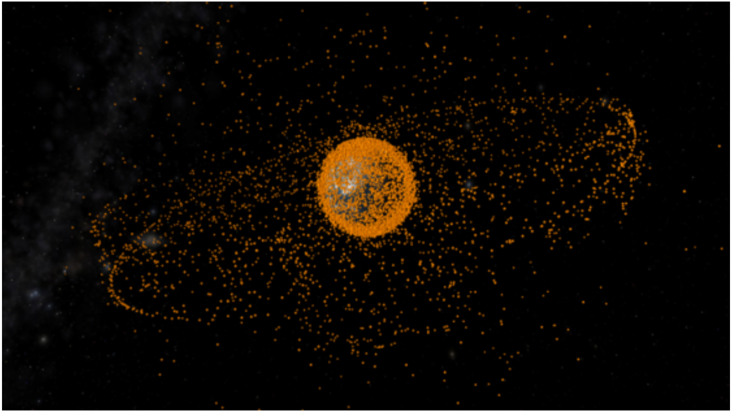
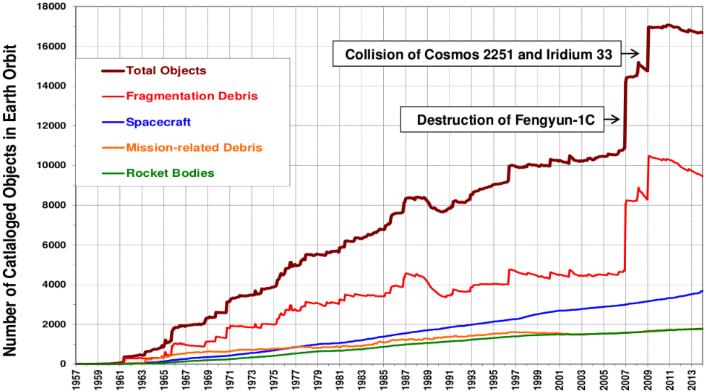
Overview

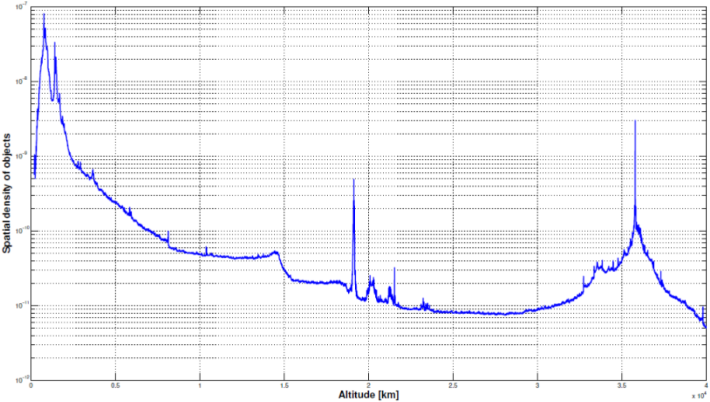
The problem of space debris



Number of catalogued space debris



Density of debris in different regions



HAMR objects

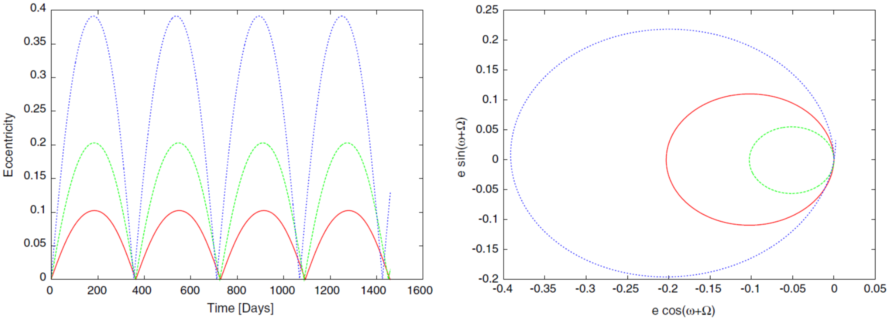
- High Area-to-Mass Ratio (HAMR) objects have been discovered in the early 2000s (Schildknecht et al.)
- Coming from thermal insulation layers (MLI) wrapping certain components of satellites that could have been detached from defunct satellite breakups, or from impact by smaller debris on said satellites
- Area-to-mass ratio ($\frac{A}{m}$) can be thousands of times higher than regular satellites : $0.01\ m^2/kg \rightarrow 30\ m^2/kg$
- HAMR objects can reach very high eccentricity (up to 0.7) in a few months and this may lead to reentry
- *Short terms* and *secular* effects appear on the eccentricity and inclination

Research and topic overview

- Analytical modeling of the GEO region via the application of canonical pertubation theory, to give insights in the dynamical evolution of objects in this region over long time scales with applications to space debris
- A specific focus is set on the recently discovered debris with high area to mass ratios since they exhibit peculiar dynamical behavior
- In this work :
  - The true nature of the forced equilibrium for space debris is shown.
  - Analytical formulas for elements describing the motion of objects at GEO (valid for HAMR objects)
  - Derivation and final expression using cylindrical coordinates (no expansion using eccentricity and inclination function)

State of the art

S. Valk et al. / Advances in Space Research 41 (2008) 1077–1090

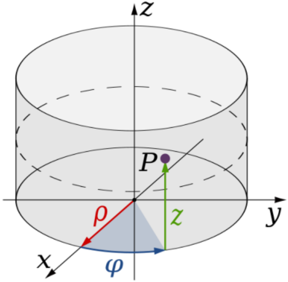


Hamiltonian of the system

- Hamiltonian $$H = T + V$$

- Kinetic energy $$T = \frac{1}{2} m \left(\dot{\rho}^2 + \rho^2 \dot{\Phi}^2 + \dot{z}^2 \right)$$

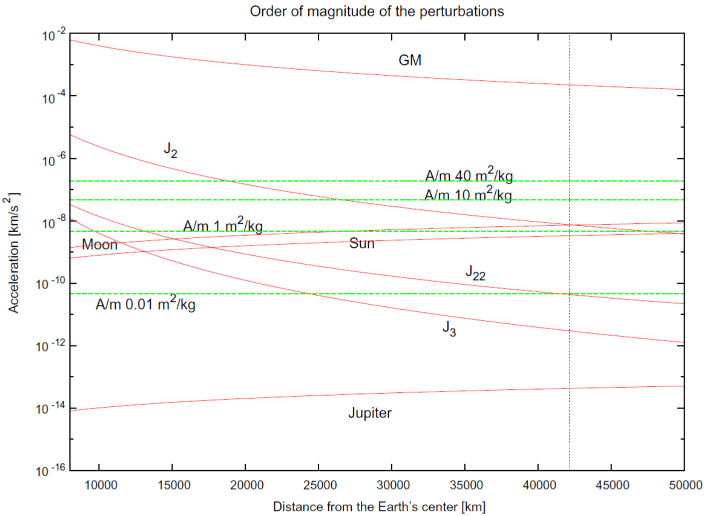
- Conjugated momenta
$$p_\rho = m\dot{\rho}$$
$$p_\Phi = m\rho^2 \dot{\Phi}$$
$$p_z = m\dot{z}$$



$\Phi = \varphi + \Omega_\oplus t$
longitude in the non-rotating frame
$\Omega_\oplus = 7.292115 \times 10^{-5}$rad/s

$$H(\rho, \Phi, z, p_\rho, p_\Phi, p_z, t) = \frac{{p_\rho}^2}{2} + \frac{{p_\Phi}^2}{2\rho^2} + \frac{p_z^2}{2} + V(\rho, \Phi, z, t)$$

Perturbations at GEO



Forces modeled

- Geopotential : up to order and degree 2
- Sun : at order 2 in the small $\left|\frac{\mathbf{r}}{\mathbf{r}_\odot}\right|$ ratio $\left(\left|\frac{\mathbf{r}}{\mathbf{r}_\odot}\right| \simeq 0.00024\right)$, eccentricity ($e \simeq 0.016709$) and inclination ($i \simeq 23.44$ deg)
- Moon : at order 2 in the small $\left|\frac{\mathbf{r}}{\mathbf{r}_☾}\right|$ ratio $\left(\left|\frac{\mathbf{r}}{\mathbf{r}_☾}\right| \simeq 0.094\right)$, eccentricity ($e \simeq 0.0055$) and inclination ($i \simeq 5.1$ deg)
- Solar radiation pressure : same as the Sun since directly dependent

Gravitational potential of the Earth

$$V_{GEO_2} = V(\rho, \varphi, z) =$$

$$- \frac{\mu_\oplus}{\sqrt{\rho^2 + z^2}} + \frac{\sqrt{5}\,\bar{C}_{2,0}\mu_\oplus R_\oplus^2}{2(\rho^2 + z^2)^{3/2}} - \frac{3\sqrt{5}\,\bar{C}_{2,0}\mu_\oplus R_\oplus^2 z^2}{2(\rho^2 + z^2)^{3/2}} + \frac{\sqrt{15}\,\bar{C}_{2,2}\mu_\oplus R_\oplus^2 z^2 \cos(2\varphi)}{2(\rho^2 + z^2)^{5/2}}$$

$$- \frac{\sqrt{15}\,\bar{C}_{2,2}\mu_\oplus R_\oplus^2 \cos(2\varphi)}{2(\rho^2 + z^2)^{3/2}} + \frac{\sqrt{15}\mu_\oplus R_\oplus^2 \bar{S}_{2,2} z^2 \sin(2\varphi)}{2(\rho^2 + z^2)^{5/2}} - \frac{\sqrt{15}\mu_\oplus R_\oplus^2 \bar{S}_{2,2} \sin(2\varphi)}{2(\rho^2 + z^2)^{3/2}}$$

Study of the system with just the Geopotential reveals the existence of *four basic equilibria* of the system, located at $\rho = \rho_{GEO}$, $z = 0$:

Stable

- $\varphi = 75.07\ deg$
- $\varphi = -104.93\ deg$

Unstable

- $\varphi = 165.07\ deg$
- $\varphi = -14.93\ deg$

Lunisolar potential

$$V_{\leftmoon} = -Gm_{\leftmoon} \left(\frac{1}{|\mathbf{r} - \mathbf{r}_{\leftmoon}|} + \frac{\mathbf{r} \cdot \mathbf{r}_{\leftmoon}}{|\mathbf{r}_{\leftmoon}|^3} \right)$$

$$V_{\odot} = -Gm_{\odot} \left(\frac{1}{|\mathbf{r} - \mathbf{r}_{\odot}|} + \frac{\mathbf{r} \cdot \mathbf{r}_{\odot}}{|\mathbf{r}_{\odot}|^3} \right)$$

Lunisolar potential

$$V_{☾} = -Gm_{☾}\left(\frac{1}{|\mathbf{r}-\mathbf{r}_{☾}|}+\frac{\mathbf{r}\cdot\mathbf{r}_{☾}}{|\mathbf{r}_{☾}|^3}\right)$$

$$V_{\odot} = -Gm_{\odot}\left(\frac{1}{|\mathbf{r}-\mathbf{r}_{\odot}|}+\frac{\mathbf{r}\cdot\mathbf{r}_{\odot}}{|\mathbf{r}_{\odot}|^3}\right)$$

$$\mathbf{r} = \begin{pmatrix} \rho\cos\Phi \\ \rho\sin\Phi \\ z \end{pmatrix} = \begin{pmatrix} \rho\cos(\varphi+\varphi_E) \\ \rho\sin(\varphi+\varphi_E) \\ z \end{pmatrix}$$

$$\varphi_E = \Omega_{\oplus}\cdot t$$

$$\Omega_{\oplus} = 7.292115\times 10^{-5}\mathrm{rad/s}$$

Associated period: 1 day

Sun and Moon vector expansions

$$r_\odot, r \cdot r_\odot = f(M_\odot)$$

- $M_\odot$: Sun's mean anomaly

$$M_\odot = f(\varphi_M)$$
$$\varphi_M = \Omega_M \cdot t$$

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$\Omega_M = 35999^\circ.049/\text{centuries}$, associated period: 1 year

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- $M_{\odot}$: Sun's mean anomaly

$$M_{\odot} = f(\varphi_M)$$

$$\varphi_M = \Omega_M \cdot t$$

$$\Omega_M = 35999^{\circ}.049/\text{centuries},$$

associated period: 1 year

$$r_{☾}, r \cdot r_{☾} = f(L_0, l, l', F_{☾}, D_{☾})$$

- L_0: Moon's mean longitude
- $l_{☾}$: Moon's mean anomaly
- $l'_{☾}$: Sun's mean anomaly
- $F_{☾}$: mean angular distance of the Moon from the ascending node
- $D_{☾}$: difference between the mean longitudes of the Sun and the Moon

$$(L_0, l, l', F_{☾}, D_{☾}) = f(\varphi_M, \varphi_{M_p}, \varphi_{M_a}, \varphi_{M_S})$$

$$\varphi_{M_a} = \Omega_{M_a}\, t$$

$$\varphi_{M_p} = \Omega_{M_p}\, t$$

$$\varphi_{M_S} = \Omega_{M_s}\, t$$

Sun and Moon vector expansions

$$r_\odot, r \cdot r_\odot = f(M_\odot)$$

- $M_\odot$: Sun's mean anomaly

$$M_\odot = f(\varphi_M)$$
$$\varphi_M = \Omega_M \cdot t$$
$$\Omega_M = 35999^\circ.049/\text{centuries},$$
associated period: 1 year

$$\Omega_{M_a} = 477198^\circ.86753/\text{centuries},$$
associated period $\sim$ 1 month
$$\Omega_{M_p} = 4069^\circ.01335/\text{centuries},$$
associated period $\sim$ 8.85 years
$$\Omega_{M_S} = 1934^\circ.13784/\text{centuries},$$
associated period $\sim$ 18.6 years

$$r_☾, r \cdot r_☾ = f(L_0, l, l', F_☾, D_☾)$$

- L_0: Moon's mean longitude
- $l_☾$: Moon's mean anomaly
- $l'_☾$: Sun's mean anomaly
- $F_☾$: mean angular distance of the Moon from the ascending node
- $D_☾$: difference between the mean longitudes of the Sun and the Moon

$$(L_0, l, l', F_☾, D_☾) = f(\varphi_M, \varphi_{M_p}, \varphi_{M_a}, \varphi_{M_S})$$
$$\varphi_{M_a} = \Omega_{M_a}\, t$$
$$\varphi_{M_p} = \Omega_{M_p}\, t$$
$$\varphi_{M_S} = \Omega_{M_s}\, t$$

Solar radiation pressure potential

$$V_{SRP} = C_r P_r \; AU^2 \; \frac{A}{m} \; \frac{1}{|\mathbf{r} - \mathbf{r}_\odot|}$$

- C_r: reflectivity coefficient $(1 \leq C_r \leq 2)$, here $C_r = 1$
- $P_r = 4,56 \times 10^{-6} \; N/m^2$: radiation pressure for an object located at AU the astronomical unit of distance
- $\frac{A}{m}$: the area-to-mass ratio
- and the same expressions are used for $\mathbf{r}_\odot$ than for the Sun's potential

Full Hamiltonian model

$$
\begin{aligned}
H &= T + V_{GEO} + V_{\odot} + V_{\leftmoon} + V_{SRP} \\
&= H(\rho, \varphi, z, \varphi_E, \varphi_M, \varphi_{M_a}, \varphi_{M_p}, \varphi_{M_S}, p_\rho, p_\varphi, p_z, J_E, J_M, J_{M_a}, J_{M_p}, J_{M_S})
\end{aligned}
$$

Full Hamiltonian model

$$
\begin{aligned}
H &= T + V_{GEO} + V_{\odot} + V_{☾} + V_{SRP} \\
&= H(\rho, \varphi, z, \varphi_E, \varphi_M, \varphi_{M_a}, \varphi_{M_p}, \varphi_{M_S}, p_\rho, p_\varphi, p_z, J_E, J_M, J_{M_a}, J_{M_p}, J_{M_S}) \\
&= \frac{{p_\rho}^2}{2} + \frac{{p_\varphi}^2}{2r^2 \sin^2\theta} + \frac{p_z^2}{2} + V(\rho, \varphi, z, \varphi_E, \varphi_M, \varphi_{M_a}, \varphi_{M_p}, \varphi_{M_S}) \\
&- \Omega_\oplus p_\varphi + \Omega_\oplus J_E + \Omega_M J_M + \Omega_{M_a} J_{M_a} + \Omega_{M_p} J_{M_p} + \Omega_{M_S} J_{M_S}
\end{aligned}
$$

Expansions and change of variables

$$H(\rho, \varphi, z, \varphi_E, \varphi_M, \varphi_{M_a}, \varphi_{M_p}, \varphi_{M_S}, p_\rho, p_\varphi, p_z, J_E, J_M, J_{M_a}, J_{M_p}, J_{M_S})$$

- 8 DOF system, 5 DOF for the motion of Sun+Moon instead of just t $\rightarrow$ 5 new frequencies in the system $(\Omega_\oplus, \Omega_M, \Omega_{M_a}, \Omega_{M_p}, \Omega_{M_S})$
- Expansion of this Hamiltonian around $\rho = \rho_{GEO}$ and $z = 0$
- $d\rho = \rho - \rho_{GEO}$, $dz = z$, and $J_\varphi = p_\varphi - p_{GEO}$ with $p_{GEO} = \Omega_\oplus \rho_{GEO}^2$
- Expansion up to order 8 in $d\rho$ and dz
- numbers of monomials in the Hamiltonian $\sim 300 \rightarrow \sim 600$

Epicyclic action-angles variables

$$d\rho = \sqrt{\frac{2J_\rho}{\kappa_\rho}} \sin(\varphi_\rho)$$
$$dz = \sqrt{\frac{2J_z}{\kappa_z}} \sin(\varphi_z)$$
$$p_\rho = \sqrt{2\kappa_\rho J_\rho} \cos(\varphi_\rho)$$
$$p_z = \sqrt{2J_z\kappa_z} \cos(\varphi_z)$$

with

$$\kappa_\rho = \sqrt{\frac{d^2 V_{GEO_{eff}}}{d\rho^2}}$$
$$\kappa_z = \sqrt{\frac{d^2 V_{GEO_{eff}}}{dz^2}}$$

where

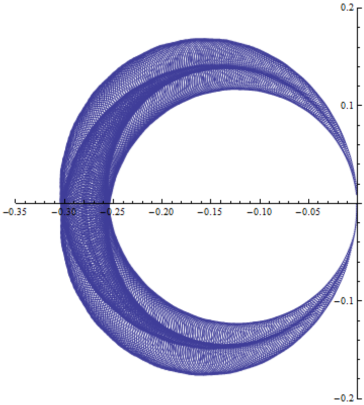
$$V_{GEO_{eff}} = \frac{p_{GEO}^2}{2r^2} + V_{GEO_0}$$
$$V_{GEO_0}(r) = -\frac{\mu_\oplus}{r}$$

The Hamiltonian now contains terms of the following form, $(m_i, k_i) \in \mathbb{Z}$:

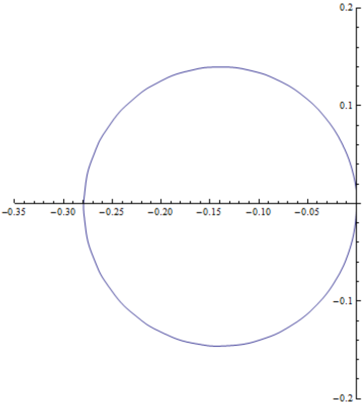
$$p_\rho^{m_1} p_\varphi^{m_2} p_z^{m_3} J_E^{m_4} J_M^{m_5} J_{M_a}^{m_6} J_{M_p}^{m_7} J_{M_S}^{m_8} \cos(k_1\varphi_\rho + k_2\varphi + k_3\varphi_z + k_4\varphi_E + k_5\varphi_M + k_6\varphi_{Ma} + k_7\varphi_{Mp} + k_8\varphi_{Ms})$$

A taste of normalization

Before



After



Normalization process

- Procedure of canonical perturbation theory
  - Around a given region of phase space (here GEO), through canonical transformations, find a Hamiltonian of the kind :
  - $H = Z + R$ where Z is 'simple' to analyze and R, the remainder is of a smaller order and induces only minor modifications to the dynamics
- *Recursive normalization algorithm*, to refine H
  - Define which terms should be allowed in Z (definition of the *module*):
    - e.g. have the form of a pendulum, or harmonic oscillator at order 0.
    - Then Z will be easier to analyze
  - $H = H_0 + \lambda H_1 + \lambda^2 H_2 + \lambda^3 H_3 ...$
  - $H^{(r)} = Z_0 + \lambda Z_1 + \lambda^2 Z_2 + ... + \lambda^{r+1} H_{r+1}^{(r)}$

Normalization process

- Canonical transformations made through the use of Lie series :
  - $(\varphi, J_\varphi) \to (\varphi^{new}, J_\varphi^{new})$ via $\chi(\varphi^{new}, J_\varphi^{new})$
  - $H^{new} = exp(L_\chi)H$ with $L_\chi = \{., \chi\}$ and $\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$
  - $exp(L_\chi) = \sum_{i=1}^{\infty} \frac{1}{k!} L_\chi^k$
    Computed easily (sum, products of derivatives) up to a given order
  - The process is easy to implement :
    - Replace the old variables by the new ones
    - Compute $H^{new} = exp(L_\chi)H$(new variables)
    - no need to find inverse functions and to use compositions
- How to find χ giving the canonical transformation that we want?
  - By solving the *homological equation* $\{Z, \chi\} = -\tilde{H}$, where $\tilde{H}$ represents the terms *not* belonging to the module.
- To pass back to the original variables: $(\varphi, J_\varphi) = exp(L_\chi)(\varphi^{new}, J_\varphi^{new})$

First Normalization

- Before normalization, the Hamiltonian contains terms with trigonometric arguments of the form:
  $k_1\varphi_\rho + k_2\varphi + k_3\varphi_z + k_4\varphi_E + k_5\varphi_M + k_6\varphi_{Ma} + k_7\varphi_{Mp} + k_8\varphi_{Ms}$
- Since *slow angles* are associated with *resonances*, we want to isolate them and construct a pendulum-like model for them
- Choice of *resonant module* to keep in Z the resonant terms: Since $\kappa_\rho \approx \kappa_z \approx \Omega_\oplus$, if $k_1 + k_3 + k_4 = 0$ then the term belongs to the module. All other terms will be relegated in the remainder through the normal form process
- This step is similar to the first step in *averaging theory*, where the mean anomaly with a daily frequency is eliminated

Change of variable after first normalization

- After this first normalization that was done up to order 4 in book-keeping we do a change of variables to reflect the slow and fast variables better

$$\varphi_{ec} = \varphi_\rho - \varphi - \varphi_E + \varphi_M \qquad J_{ec} = J_\rho$$
$$\varphi_R = \varphi \qquad J_R = J_\varphi + J_{ec} + J_{in}$$
$$\varphi_{in} = \varphi_z - \varphi - \varphi_E \qquad J_{in} = J_z$$
$$\varphi_e = \varphi_E \qquad J_e = J_E + J_\rho + J_z$$
$$\varphi_m = \varphi_M \qquad J_m = J_{\varphi_M} - J_\rho$$
$$\varphi_{ma} = \varphi_{Ma} \qquad J_{ma} = J_{\varphi_{Ma}}$$
$$\varphi_{mp} = \varphi_{Mp} \qquad J_{mp} = J_{\varphi_{Mp}}$$
$$\varphi_{ms} = \varphi_{Ms} \qquad J_{ms} = J_{\varphi_{Ms}}$$

Introduction of Poincaré variables

- We can see in the Hamiltonian obtained by the previous procedure some terms showing the form a forced equilibrium, notably in the variables J_{ec} and J_{in}
- It is then natural to pass to Poincaré variables:

$$
\begin{aligned}
x_e &= \sqrt{2J_{ec}}\sin(\varphi_{ec})\\
y_e &= \sqrt{2J_{ec}}\cos(\varphi_{ec})\\
x_i &= \sqrt{2J_{in}}\sin(\varphi_{in})\\
y_i &= \sqrt{2J_{in}}\cos(\varphi_{in})
\end{aligned}
$$

Forced equilibrium

- By removing all the other terms than (x_e, y_e, x_i, y_i), we have a toy model. We can then look for its equilibrium
- The solutions found are :

$$
\begin{aligned}
x_{ef} &= -0.138785 \\
y_{ef} &= -0.000012 \\
x_{if} &= -0.231519 \\
y_{if} &= 0.042744
\end{aligned}
$$

- Which corresponds to (for $\frac{A}{m} = 10$ for instance):
  - a forced eccentricity of : 0.11
  - a forced inclination of : 10.9 deg

Expansion around the forced equilibrium

- Numerically integrating this Hamiltonian starting at the equilibrium found gives variations of up to 10% for (x_e, y_e, x_i, y_i). This shows the need for a refined equilibrium
- To find it we expand our Hamiltonian around $(x_{ef}, y_{ef}, x_{if}, y_{if})$ by introducing new variables:

$$dx_e = x_e - x_{ef}$$
$$dy_e = y_e - y_{ef}$$
$$dx_i = x_i - x_{if}$$
$$dy_i = y_i - y_{if}$$

Transformation to action-angle variables

- Need to pass to action-angle variables again, for normalization purposes
- First, a diagonalization of the previous Hamiltonian toy model restricted to its quadratic terms is done $(dx_e, dy_e, dx_i, dy_i) \to (q_1, p_1, q_2, p_2)$
- Then to finish the transformation to action-angle variables we do the following canonical transformation:

$$q_k \to \sqrt{J_k} e^{i\varphi_k}$$
$$p_k \to -i\sqrt{J_k} e^{-i\varphi_k}$$

- To refine the equilibrium, a new normalization of this Hamiltonian is needed

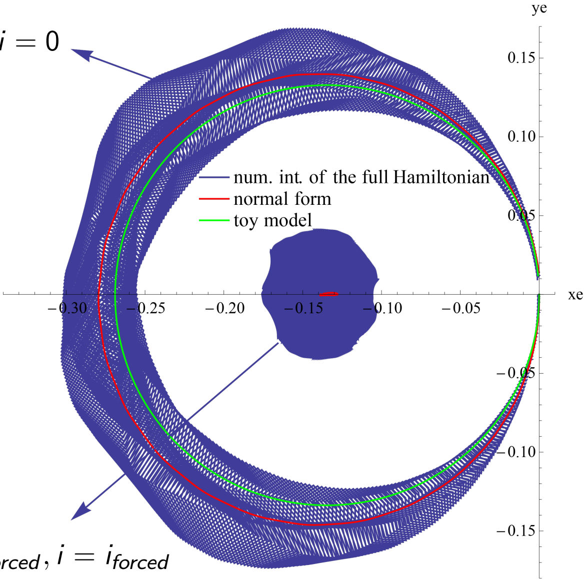
Second normalization

- This time, we want to eliminate very precise terms, the terms that cause the non-constancy of (dx_e, dy_e, dx_i, dy_i)
- They are terms linear in (dx_e, dy_e, dx_i, dy_i) and contain *small divisors*
- Define a threshold up to which we keep the terms with a given small divisor so that they contribute to the dynamics of the normal form
- New normalization will eliminate the other ones up to second order in book-keeping

Nature of the forced equilibrium

- Back-transforming from the action-angle variables $(J_1, \varphi_1, J_2, \varphi_2)$ with which the Hamiltonian was normalized to the (dx_e, dy_e, dx_i, dy_i), and numerically integrating the results, the variations are smaller by one order of magnitude
- We now consider those variations small enough, and therefore the new (dx_e, dy_e, dx_i, dy_i) as quasi-constants. We can then express the old (dx_e, dy_e, dx_i, dy_i) in function of these new ones considered as constant (equal to their refined equilibrium values) and have an expression of them directly function of time
- The forced equilibrium is then a lower dimensional object containing a combination of five distinct frequencies $(\Omega_{\oplus}, \Omega_M, \Omega_{M_a}, \Omega_{M_p}, \Omega_{M_S})$

$e = 0, i = 0$



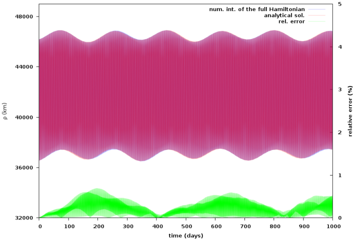
$e = e_{forced}, i = i_{forced}$

Analytical expression for the original variables

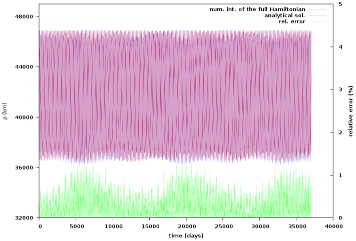
- From these original (dx_e, dy_e, dx_i, dy_i) expressed in function of time we can come back to the (ρ, φ, z) and express them in function of time only too
- This gives us an analytical formula of our original variables

Comparison of the analytical results with the numerical integration of the full model

ρ for 1000 days

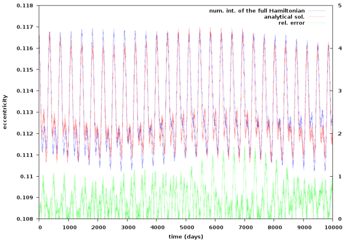


ρ for 100 years

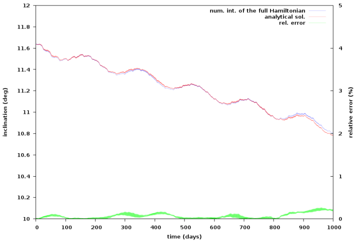


Comparison of the analytical results with the numerical integration of the full model

eccentricity for 10000 days



inclination for 1000 days



Future works

- Do a thorough study of the region using stability maps
- Derive orbital lifetime of objects
- Dynamical deorbiting strategies
- Study other regions of the phase space

Thank you for your attention!
Questions?