An efficient code to solve the Kepler's equation for elliptic and hyperbolic orbits

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Overview: Kepler equation

Kepler equation provides the position of the object orbiting around a body for some specific time.

Elliptical orbits: $x = y - e \sin y$

Unknown parameter → y=E

Known parameters → x=M ; e

Approaches	Markley (1995)	Fukushima (1996)	Mortari & Elipe (2014)
E starter estimation	X	X	
E bounds estimation			X
Iterative method		X	X
No iterative method	X		

Hyperbolic orbits: $x = e \sinh y - y$

Unknown parameter → y=H

Known parameters → x=M_H ; e

Approaches	Gooding (1988)	Fukushima (1997)
E starter estimation	X	X
E bounds estimation		
Iterative method	X	X
No iterative method		X

Motivation

- Take advantage of the full potential of the symbolic manipulators.

- Efficient solving of Kepler equation estimating a good initial seed for the eccentric and hyperbolic anomaly:
  - * To improve the computational time
  - * To reach the machine error accuracy with hardly iterations

- Appropriate algorithm in the singular corner of the Kepler equation:
  - * Neighborhood of M =0 and e =1
  - * To avoid convergence problems in the numerical method

- The advantage of the good behavior of the modified Newton-Raphson method when the initial seed is close to the looked for solution.

- Applicability to other problems: Lambert's problem

Code solution: The N-R methods

- Modified NR method: Solution of the equation defined by a successive approximation method starting from the seed (y_o):

$$y_{i+1} = y_i + \Delta y_i,\ i \epsilon \aleph \quad \rightarrow \quad f(y_{i+1}) = f(y_i + \Delta y_i) = 0$$

Second order Taylor expansion about y_i : $f(y_{i+1}) \approx f(y_i) + f'(y_i)\Delta y_i + \frac{1}{2} f''(y_i)\Delta {y_i}^2 = 0$

$$\Delta y_{i \approx} \frac{-f'(y_i) + \sqrt{f'(y_i)^2 - 2f(y_i)f''(y_i)}}{f''(y_i)} = -\frac{2f(y_i)}{f'(y_i) \pm \sqrt{|f'(y_i)^2 - 2f(y_i)f''(y_i)|}} \qquad \begin{matrix} + & \rightarrow & f'(y_i) > 0 \\ - & \rightarrow & f'(y_i) < 0 \end{matrix}$$

- Generalization of the modified N-R method → Root-finding method of Laguerre (Conway 1986)

$$\Delta y_{i \approx} - \frac{nf(y_i)}{f'(y_i) \pm \sqrt{|(n-1)[(n-1)f'(y_i)^2 - nf(y_i)f''(y_i)]|}}$$

$f(y_i) \equiv 0$
Kepler equation for elliptic or hyperbolic orbit

Classical N-R method	Modified N-R method	Conway method
n = 1	n = 2	n = 5

Elliptic Kepler equation

$$x = y - e \, sin \, y$$

Code solution: The seed value I

Steps:

1. The E-domain $[0, \pi]$ is discretized in 12 intervals of 15º of longitude :

$$E_i = \frac{(i-1)\pi}{12} \quad i = 1, \dots 13$$

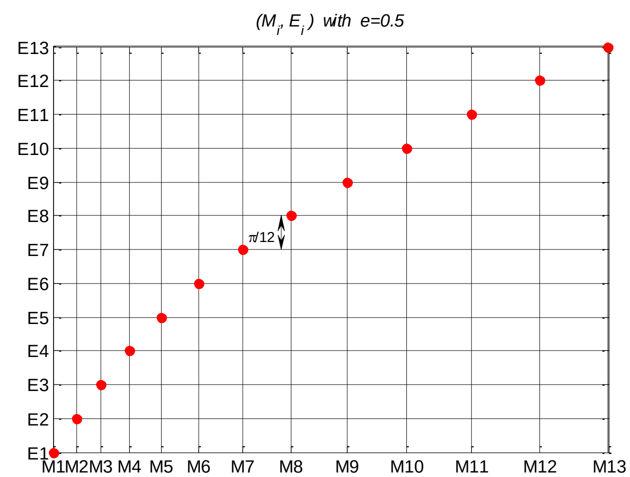
M1	E1	M13	E13
0	0	π	π

2. The M-domain is discretized according to the Kepler equation

$$M_i = E_i - e\sin E_i \quad i = 1, \dots 13$$

* If $x = M > \pi \rightarrow M = 2\pi - \chi \,;\, E = 2\pi - \eta$

3. For each interval a fifth degree polynomial $p_i(x)$ is defined to interpolate the eccentric anomaly.



$$M \in [M_i, M_{i+1}] \rightarrow [E_i, E_{i+1}] \qquad with \quad i = 1, \dots, 12$$

$$\boldsymbol{p_i(x) = a_o^i + a_1^i x + a_2^i x^2 + a_3^i x^3 + a_4^i x^4 + a_5^i x^5}$$

Code solution: The seed value II

4. Six boundary conditions are imposed to determine the coefficients of $p_i(x)$

$$\boldsymbol{p_i}(\boldsymbol{x}) = a_o^i + a_1^i x + a_2^i x^2 + a_3^i x^3 + a_4^i x^4 + a_5^i x^5 \qquad with \;\; i = 1, \dots, 12$$

The six coefficients of $p_i(x)$ are obtained by six conditions at both ends of the corresponding interval:

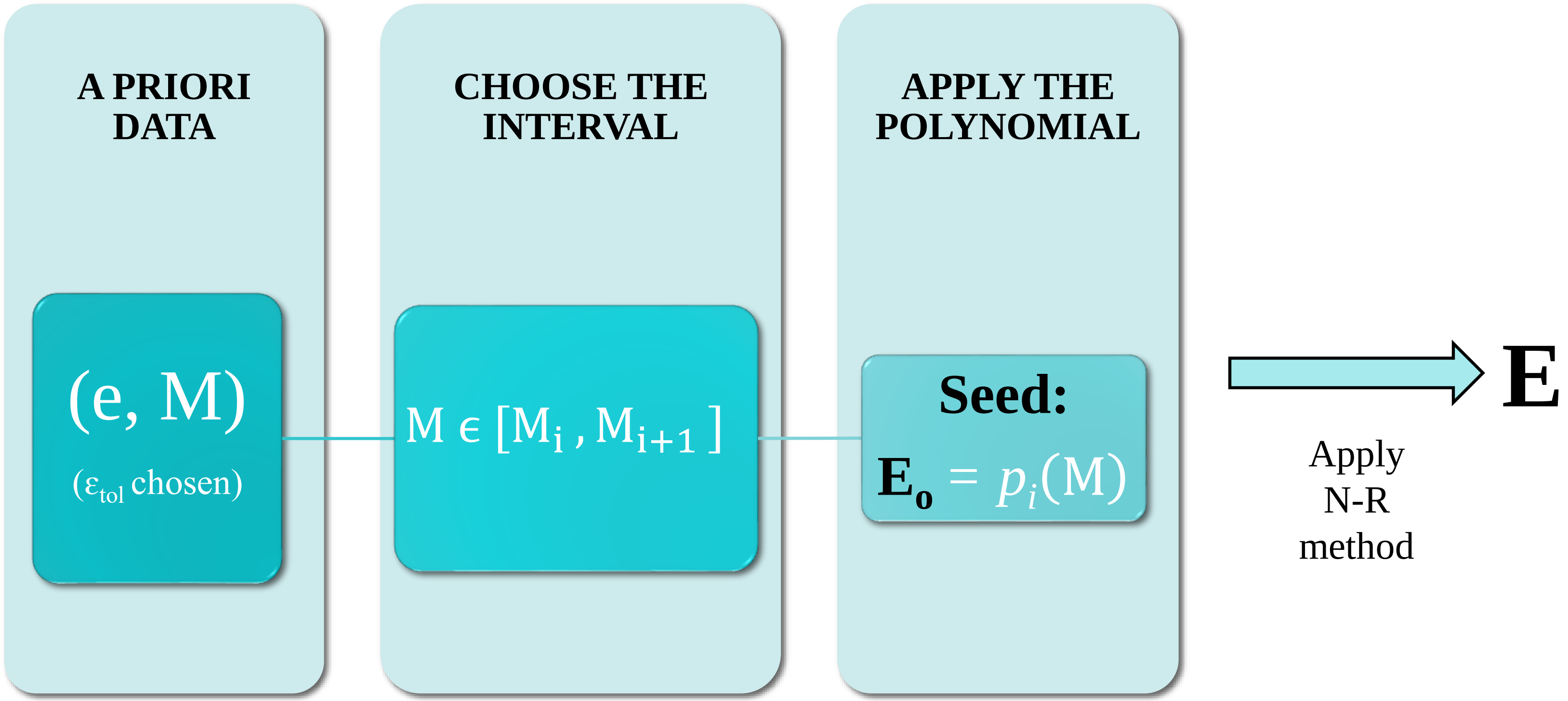
$$p_i(x_i) = \mathrm{y}(x_i) = E_i \qquad p'_i(x_{i+1}) = \mathrm{y}'(x_{i+1})$$

$$p'_i(x_i) = \mathrm{y}'(x_i) \qquad p_i(x_{i+1}) = \mathrm{y}(x_{i+1})$$

$$p''_i(x_i) = \mathrm{y}''(x_i) \qquad p''_i(x_{i+1}) = y''(x_{i+1})$$

5. Given e and M, the starting value E_o is estimated: $E_0 = p_i(x = M)$

SDG-code


A PRIORI DATA
(e, M)
(ε_{tol} chosen)
CHOOSE THE INTERVAL
$M \in [M_i , M_{i+1}]$
APPLY THE POLYNOMIAL
Seed:
$E_o = p_i(M)$
Apply N-R method
E

Analysis of the singularity I

- Problem statement: Kepler equation $y - e\sin y - x = 0$ has a singular behavior in the neighborhood of e=1 and M=0
- Goal: Describe numerically the exact solution (y_v) with enough accuracy to be part of the seed (E_0) used to start the N-R process.
- Solution: Apply an asymptotic expansion in power of the small parameter $\epsilon = 1 - e << 1$

$$\epsilon \neq 0 \rightarrow y - (1-\epsilon)\sin y - x = 0 \qquad \epsilon = 0 \rightarrow y_o - \sin y_o - x = 0$$

* Asymptotic expansion → $x = x(y_o)$
* $x(y_o)$ inverted with Maple symbolic simulator:

$$\boldsymbol{y_o(x)} = (6x)^{\frac{1}{3}} + \frac{1}{10}x + \frac{1}{1400}(6x)^{\frac{5}{3}} + \dots$$

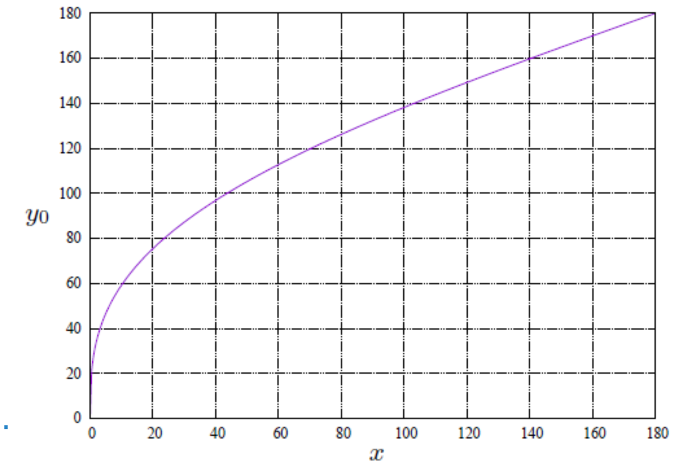
* Asymptotic expansion in the limit $\epsilon \rightarrow 0$ ($\epsilon \neq 0$):

$$\boldsymbol{y(x)} = y_o(x) + \epsilon\, y_1(x) + \epsilon^2 y_2(x) + \dots$$

$y_i(x), i = 1, \dots, n$ as a function of the known $y_o(x)$ vanishing the resulting serie for every order in ϵ :

$$\boldsymbol{y_{as}(x)} = y_o - \epsilon\frac{\sin y_o}{1-\cos y_o} + \frac{\epsilon^2}{2}\frac{\sin y_o}{1-\cos y_o} + \frac{\epsilon^3}{3}\frac{\cos y_o}{\sin y_o}\frac{(2-\cos y_o)(1+\cos y_o)}{(1-\cos y_o)^2} \dots$$

Solution $y_o(x)$ when $\epsilon = 0$



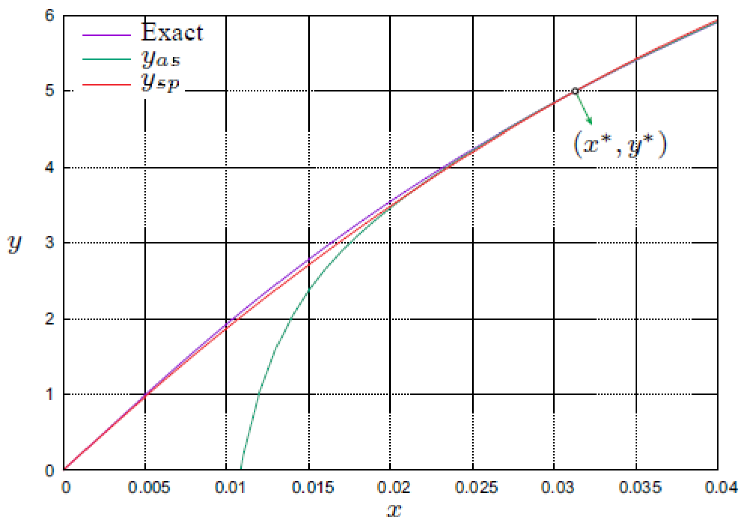
Analysis of the singularity II

- Exact solution: $y_v - (1-\epsilon)\sin y_v - x = 0 \rightarrow y_v$ (with Maple symbolic simulator)

- Asymptotic solution:

$$\begin{cases} y_o(x) = (6x)^{\frac{1}{3}} + \frac{1}{10}x + \frac{1}{1400}(6x)^{\frac{5}{3}} + \ldots \qquad x = \text{M (known)} \\ y_{as}(x) = y_o - \epsilon\frac{\sin y_o}{1-\cos y_o} - \frac{\epsilon^2}{2}\frac{\sin y_o}{1-\cos y_o} + \frac{\epsilon^3}{3}\frac{\cos y_o}{\sin y_o}\frac{(2-\cos y_o)(1+\cos y_o)}{(1-\cos y_o)^2} \ldots \end{cases}$$

Critical point when $\epsilon = 0.005$, $\varepsilon_{tol} = 0.001$



Fixed ε_{tol} : $\exists\,(x^*, y^*)$ such as $|y_v - y_{as}| \geq \varepsilon_{tol}$ for the first time

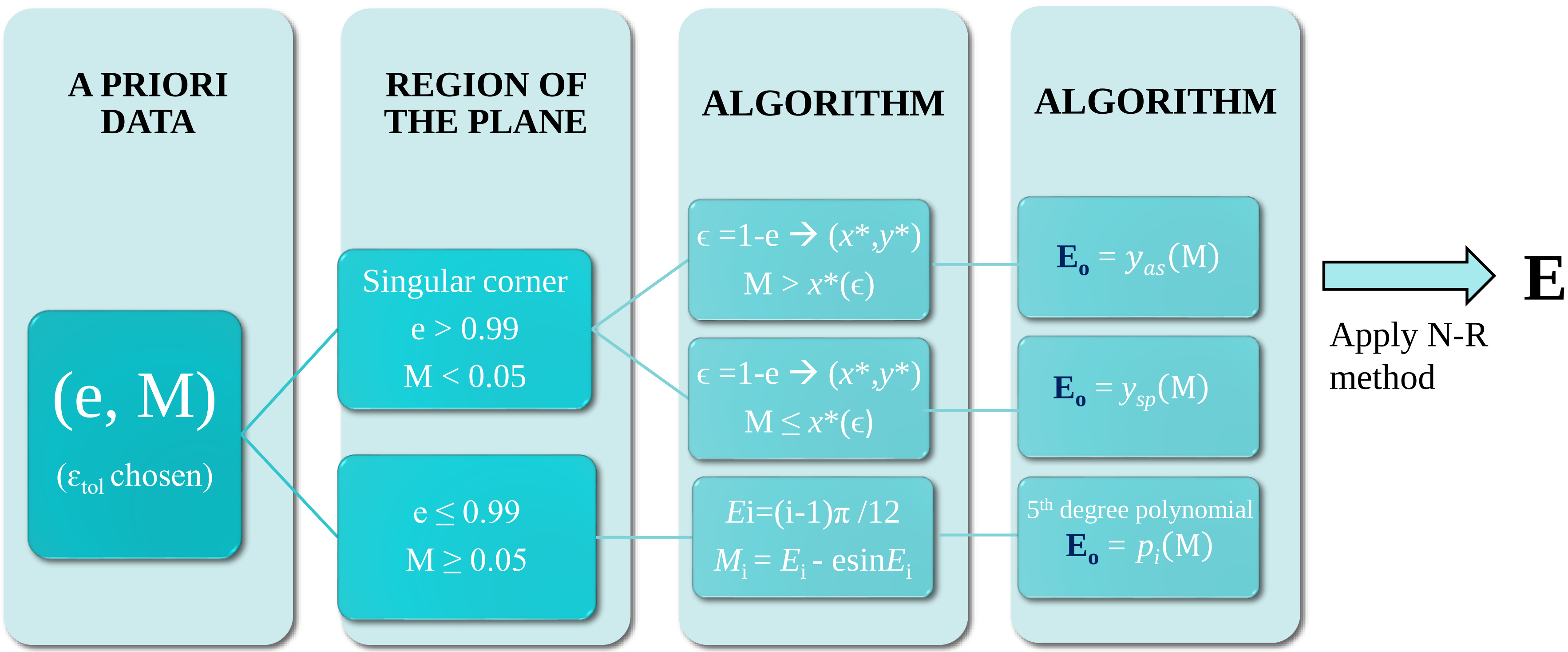
- Special solution ($x < x^*$): $y_{sp}(x) = x\,(ax + \frac{1}{\epsilon})$, $a = \frac{y^*}{(x^*)^2} - \frac{1}{\epsilon x^*}$

  * Assuming $\varepsilon_{tol} = 5\text{x}10^{-4}$, for each $\epsilon_i \rightarrow (x^*, y^*)_i$ „ ϵ defined in [0, 0.025]

  * Least Square Adjustment to fit the critical points (x^*, y^*) w.r.t. ϵ :

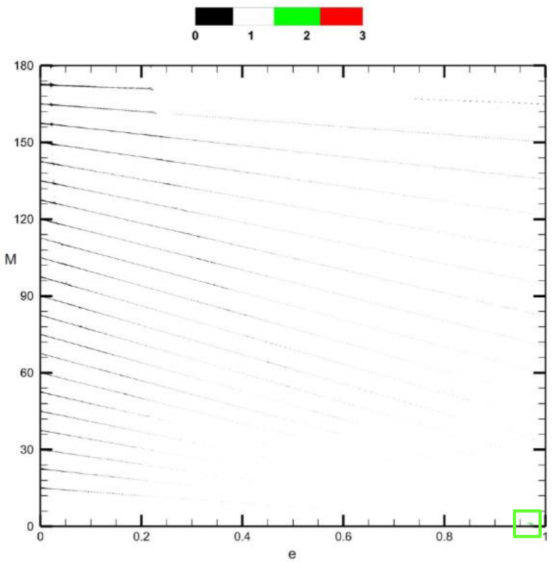
$$\begin{cases} x^* = \epsilon(-86.3921\epsilon^2 + 9.1074\epsilon + 0.051632) \\ y^* = \sqrt{\epsilon}\,(-220.1588\epsilon^2 + 12.0785\epsilon + 0.9972) \end{cases}$$

SDG-code



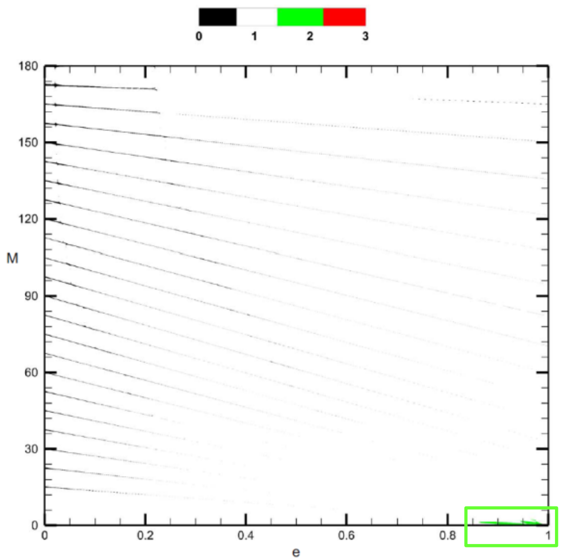
A PRIORI DATA
REGION OF THE PLANE
ALGORITHM
ALGORITHM
(e, M)
(ε_{tol} chosen)
Singular corner
e > 0.99
M < 0.05
e ≤ 0.99
M ≥ 0.05
ϵ =1-e → (x^*,y^*)
M > $x^*(\epsilon)$
ϵ =1-e → (x^*,y^*)
M ≤ $x^*(\epsilon)$
$E_i=(i-1)\pi/12$
$M_i = E_i - e\sin E_i$
$\mathbf{E_o} = y_{as}(M)$
$\mathbf{E_o} = y_{sp}(M)$
5th degree polynomial
$\mathbf{E_o} = p_i(M)$
Apply N-R method
E

Results: SDG-code using MNR, Conway and CNR method



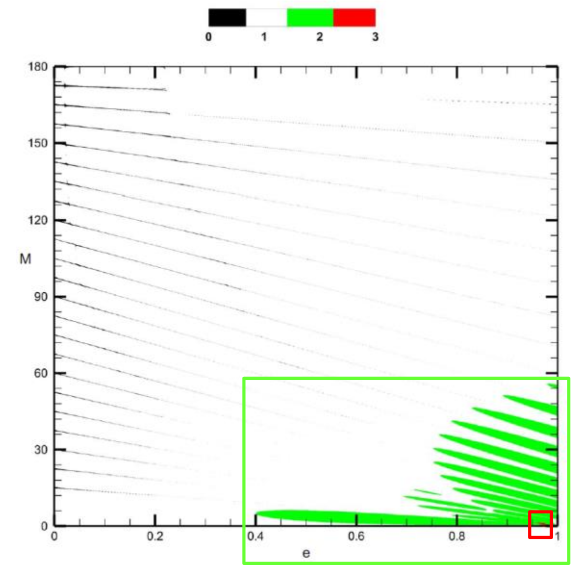
Modified N-R method:

≤ 2 iterations
$i_m \approx 0.99$
CPU time = 68.4 sec



Conway method

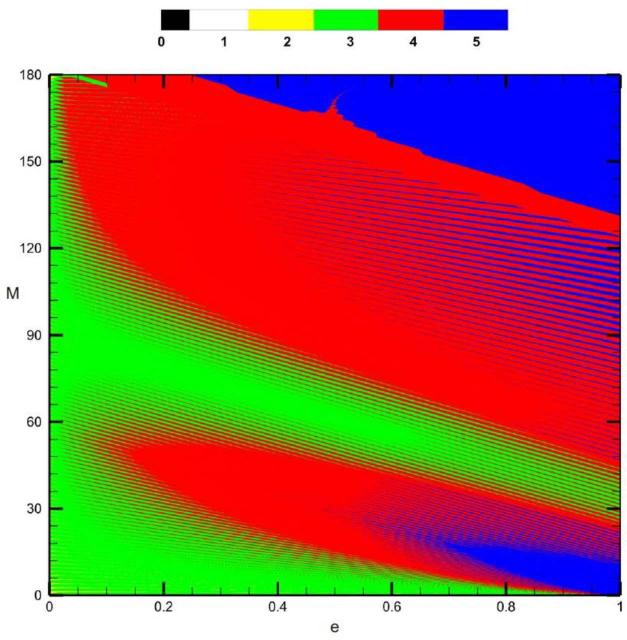
≤ 2 iterations
$i_m \approx 0.99$
CPU time = 70.4 sec

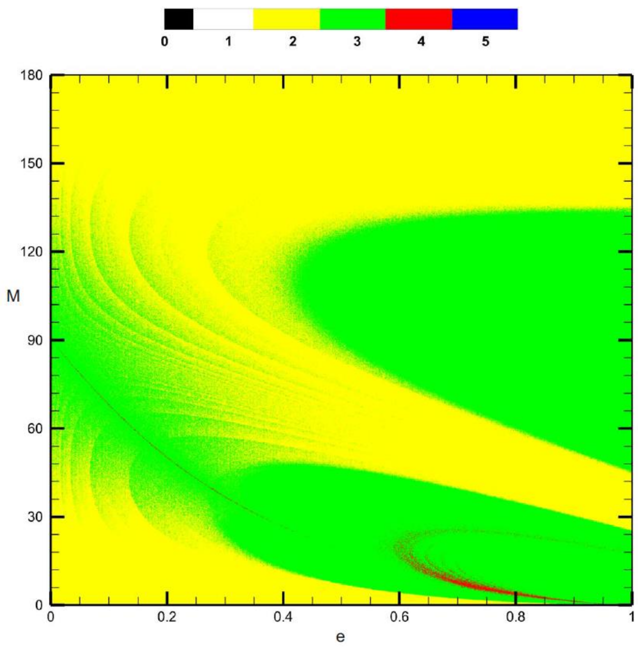


Classical N-R method

≤ 3 iterations
$i_m \approx 1.037$
CPU time = 62.1 sec

Results: Fukushima and Mortari code





Fukushima method:

$i_m \approx 4.030$
CPU time = 328 sec

Mortari method:

$i_m \approx 2.427$
CPU time = 101.4 sec

Conclusions

- An efficient code has been developed to solve the Kepler equation for elliptic motion.
- Improving the seed estimator provides faster and more accurate results than improving the numerical mehod:

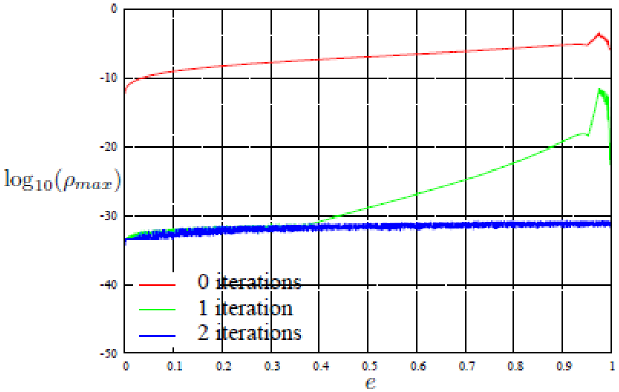
		SDG – code (MNR)	SDG – code (Conway)	SDG – code (CNR)	Fukushima code	Mortari & Elipe code
Iterations	0 (%)	0.66	0.66	0.66	0.05	0.1
	1 (%)	99.33	99.28	95.02	0	0.05
	2 (%)	0.0052	0.057	4.3	0.35	57.20
	3 (%)	0	0	0.0093	25.75	44.32
	≥ 4 (%)	0	0	0	73.85	0.33
	Mean value	0.99	0.99	1.037	4.030	2.427
Points		~ 4 millions	~ 4 millions	~ 4 millions	~ 4 millions	~ 4 millions
Computational time		68.4 sec	70.4 sec	62.1 sec	328 sec	101.4 sec

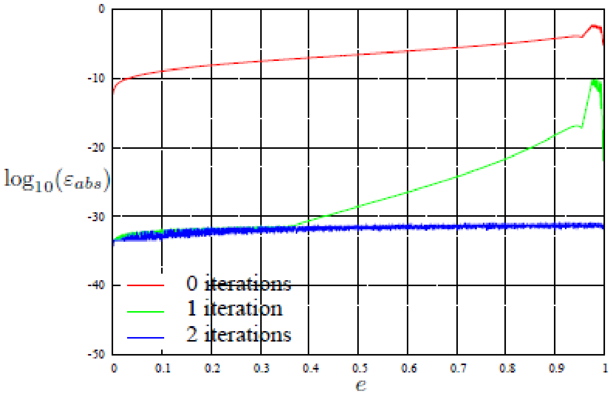
Accuracy analysis

- Considering the true solution (y_v), we study the residual (ρ) and the absolute error (ε_{abs}) of the numerical solution (y_c) taking into account that:

$$\left.\begin{array}{r} y_c = y_v + \varepsilon_{abs} \\ \rho = |y_c - e\sin y_c - x| \end{array}\right\} \quad |\varepsilon_{abs}| = \frac{\rho}{|1-\cos y_v|}$$

- Fixing the value of the eccentricity, we scan the whole interval $M \in [0, \pi]$ and calculate the residual and absolute error after zero iteration, one iteration, two iterations and so on, considering the maximum residual that we found for each iteration when the M interval is scanning:





Hyperbolic Kepler equation

$$x = e\, sinh\, y - y$$

Code solution: The seed value I

Steps:

1. A change of variable is done in the Kepler equation:

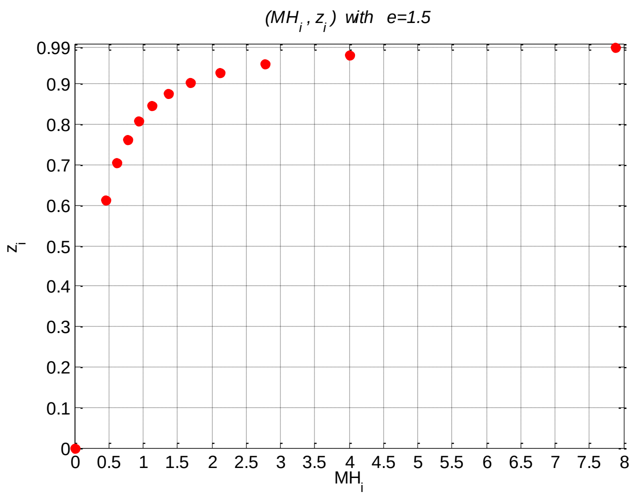
$$z = \tanh y \leftrightarrow y = \frac{1}{2}\ln\frac{1+z}{1-z} \quad \Rightarrow \quad x = e\frac{z}{\sqrt{1-z^2}} - \frac{1}{2}\ln\frac{1+z}{1-z}$$

with $z \in [0,1)$ when $y \in [0,\infty)$ and x singular in $z = 1$

M_{H1}	z_1	M_{H13}	z_{13}
0	0	∞	1

2. The z-domain [0, 1) is discretized in 12 uneven intervals such that:

$$\begin{cases} z_i = 0.99\left(\frac{i-1}{11}\right)^{\frac{1}{5}} & i = 1,\ldots 12 \\ z_{13} = 1 \end{cases}$$



3. The mean anomaly domain is discretized according to the Kepler equation $x(z)$ with respect to the first twelve z_i

Code solution: The seed value II

4. For each of the first 11 intervals, $z \in [0, 0.99]$:

*A fifth degree polynomial $p_i(x)$ is defined to interpolate the variable z.
*Six boundary conditions are imposed to determine the coefficients of $p_i(x)$ as we did for the elliptic case

$$\boldsymbol{p_i(x)} = a_o^i + a_1^i x + a_2^i x^2 + a_3^i x^3 + a_4^i x^4 + a_5^i x^5 \quad with \;\; i = 1, \dots, 12$$

such that given e and M_H, the starting value z_o is estimated: $Z_0 = p_i(x = M_H)$

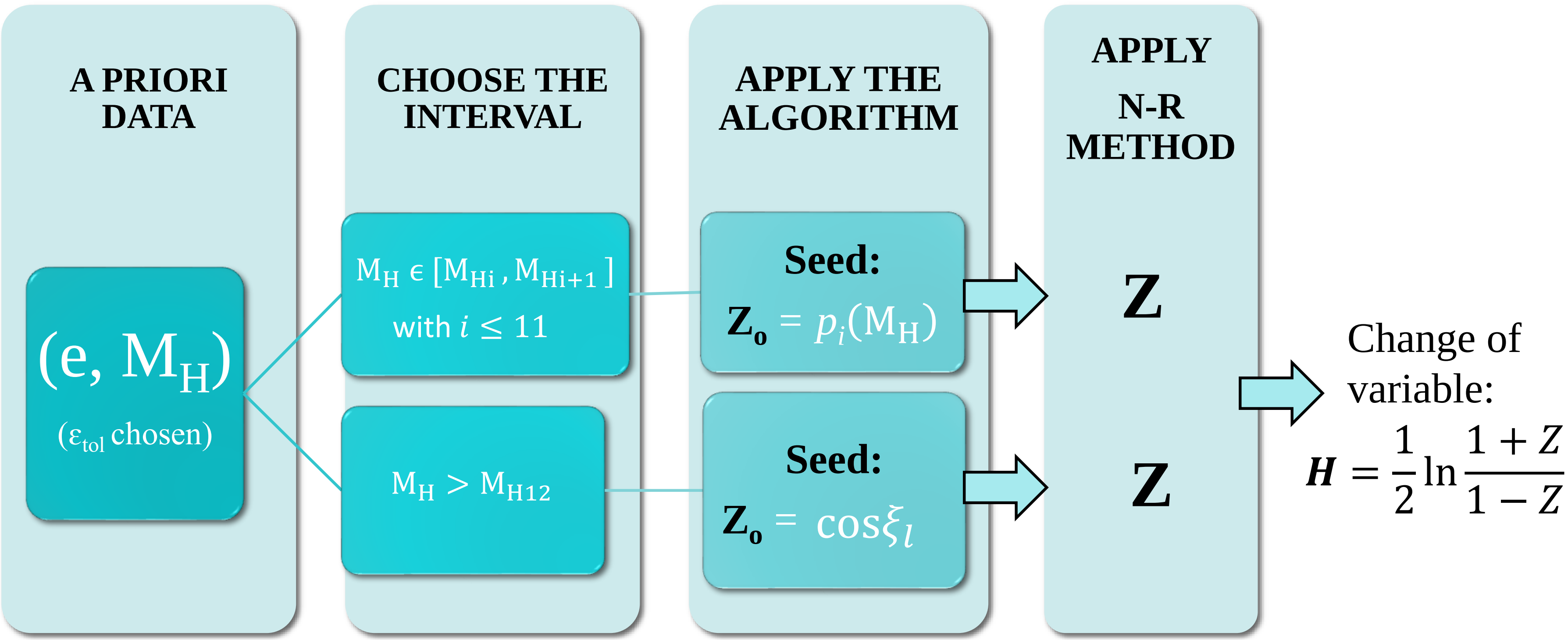
5. In the last interval, $z \in [0.99, 1)$:

A recursive algorithm is applied doing a change of variable in $x(z)$: $z = \cos\xi \;\Rightarrow\; x = e\cot\xi - \frac{1}{2}\ln\left(\cot^2\frac{\xi}{2}\right)$

$$\xi = h(\xi, e, x) = \arctan\frac{e}{x + \frac{1}{2}\ln\left(\cot^2\frac{\xi}{2}\right)} \quad \Rightarrow \quad \xi_{n+1} = h(\xi_n, e, x) \quad \text{with starter} \quad \xi_0 = \frac{\pi}{2}$$

such that given e and M_H, the starting value Z_o is estimated: $Z_0 = \cos\xi_l$

SDG-code


A PRIORI DATA
(e, M_H)
(ε_{tol} chosen)
CHOOSE THE INTERVAL
$M_H \in [M_{Hi}, M_{Hi+1}]$ with $i \leq 11$
$M_H > M_{H12}$
APPLY THE ALGORITHM
Seed: $Z_o = p_i(M_H)$
Seed: $Z_o = \cos\xi_l$
APPLY N-R METHOD
Z
Z
Change of variable:
$H = \frac{1}{2}\ln\frac{1+Z}{1-Z}$

Analysis of the singularity I

- Problem statement: Kepler equation $e\sinh y - y - x = 0$ has a singular behavior in the neighborhood of e=1 and M_H=0
- Goal: Describe numerically the exact solution (y_v) with enough accuracy to be part of the seed (H_0) used to start the N-R process.
- Solution: Apply an asymptotic expansion in power of the small parameter $\epsilon = e-1 << 1$

$$\epsilon \neq 0 \rightarrow y - (1+\epsilon)\sinh y + x = 0 \qquad \epsilon = 0 \rightarrow y_o - \sinh y_o + x = 0$$

* Asymptotic expansion → $x = x(y_o)$
* $x(y_o)$ inverted with Maple symbolic simulator:

$$\boldsymbol{y_o(x)} = 1.817121(x)^{\frac{1}{3}} - \frac{1}{10}x + 0.0141511(x)^{\frac{5}{3}} + \ldots$$

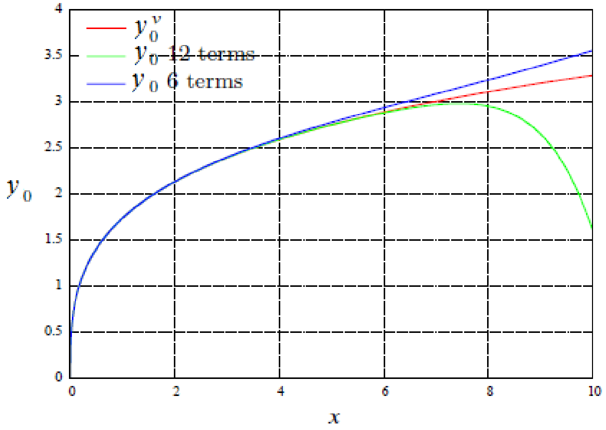
* Asymptotic expansion in the limit $\epsilon \rightarrow 0$ ($\epsilon \neq 0$):

$$\boldsymbol{y(x)} = y_0(x) + \epsilon\, y_1(x) + \epsilon^2 y_2(x) + \ldots$$

$y_i(x), i = 1, \ldots, n$ as a function of the known $y_o(x)$ vanishing the resulting serie for every order in ϵ :

$$\boldsymbol{y_{as}(x)} = y_o + \epsilon\frac{sinhy_o}{1-coshy_o} - \frac{\epsilon^2}{2}\frac{sinhy_o}{1-coshy_o} + \frac{\epsilon^3}{3}coshy_o sinhy_o\frac{(2-coshy_o)(1+coshy_o)}{(1-coshy_o)^2}$$

Solution $y_o(x)$ when ϵ =0



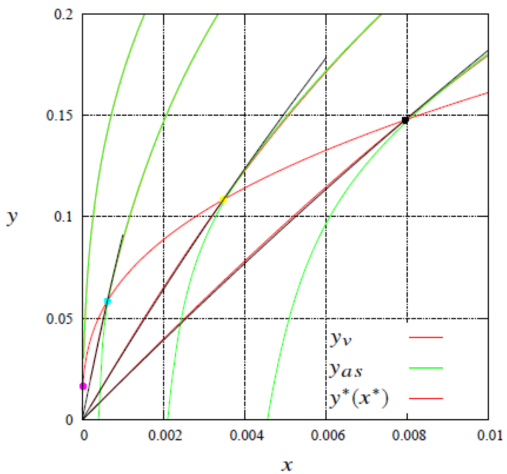
Analysis of the singularity II

- Exact solution: $y_v - (1+\epsilon)\sinh y_v + x = 0 \rightarrow y_v$ (with Maple symbolic simulator)

- Asymptotic solution:

$$\begin{cases} y_o(x) = 1.817121(x)^{\frac{1}{3}} - \frac{1}{10}x + 0.0141511(x)^{\frac{5}{3}} + \ldots \quad x = M_H \text{ (known)} \\ y_{as}(x) = y_o + \epsilon\frac{\sinh y_o}{1-\cosh y_o} - \frac{\epsilon^2}{2}\frac{\sinh y_o}{1-\cosh y_o} + \frac{\epsilon^3}{3}\cosh y_o \sinh y_o \frac{(2-\cosh y_o)(1+\cosh y_o)}{(1-\cosh y_o)^2} \ldots \end{cases}$$

Critical points when ϵ = 0.001, 0.01, 0.03, 0.05 , ε_{tol} = 1.5x10−3



Fixed ε_{tol} : $\exists$ (x^*, y^*) such as $|y_v - y_{as}| \geq \varepsilon_{tol}$ for the first time

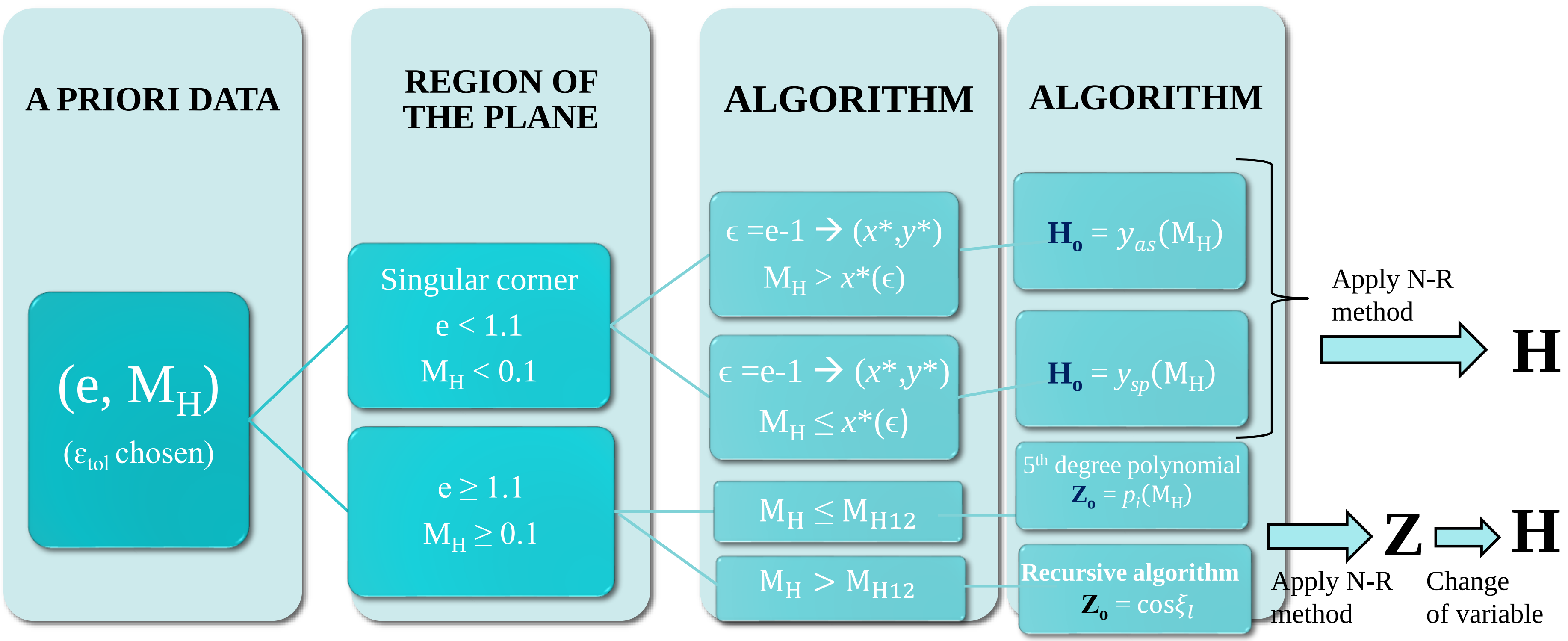
- Special solution ($x < x^*$): $y_{sp}(x) = x\,(ax + \frac{1}{\epsilon})$, $a = \frac{y^*}{(x^*)^2} - \frac{1}{\epsilon x^*}$

  * Assuming $\varepsilon_{tol} = 1.5\text{x}10^{-3}$, for each $\epsilon_i \rightarrow (x^*, y^*)_i$ „ ϵ defined in [0, 0.1]

  * Least Square Adjustment to fit the critical points (x^*, y^*) w.r.t. ϵ :

$$\begin{cases} x^* = 0.023988\epsilon + 4.300478\epsilon^2 - 62.308284\epsilon^3 + 869.10223\epsilon^4 - \ldots \\ y^* = 0.549826\sqrt{\epsilon} + 3.685319\epsilon\sqrt{\epsilon} - 53.136123\epsilon^2\sqrt{\epsilon} + \ldots \end{cases}$$

SDG-code



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(e, M_H)
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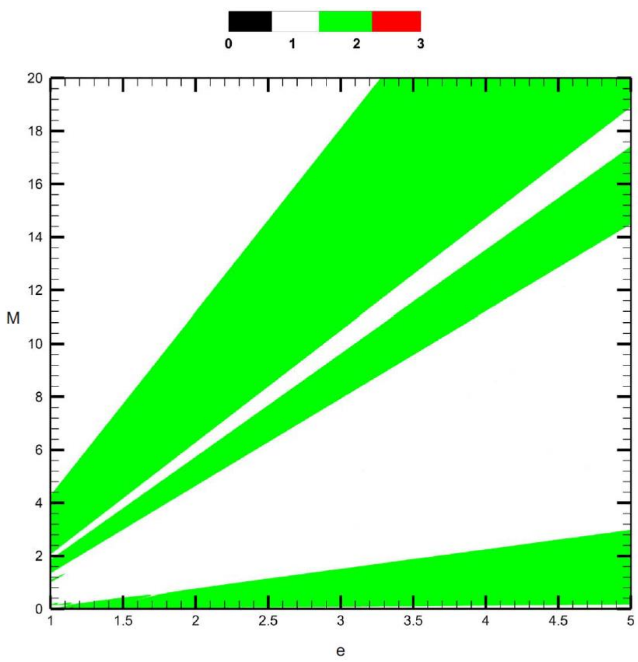
Singular corner
e < 1.1
$M_H < 0.1$
$e \geq 1.1$
$M_H \geq 0.1$
ϵ =e-1 → (x^*,y^*)
$M_H > x^*(\epsilon)$
ϵ =e-1 → (x^*,y^*)
$M_H \leq x^*(\epsilon)$
$M_H \leq M_{H12}$
$M_H > M_{H12}$
$\mathbf{H_o} = y_{as}(M_H)$
$\mathbf{H_o} = y_{sp}(M_H)$
5th degree polynomial
$\mathbf{Z_o} = p_i(M_H)$
Recursive algorithm
$\mathbf{Z_o} = \cos\xi_l$
Apply N-R method
H
Z
H
Apply N-R method
Change of variable

Preliminary results: SDG-code using the MNR method

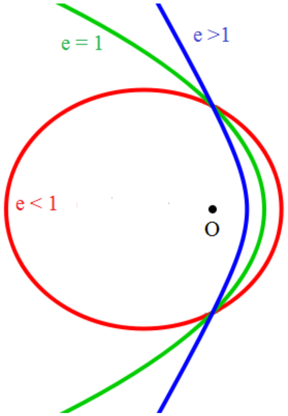


Iterations	0 (%)	0.026
	1 (%)	59.09
	2 (%)	40.88
	3 (%)	0.00072
	≥ 4 (%)	0
	Mean value	1.408
Points		~ 16 millions
Computational time		394 sec

Future Work

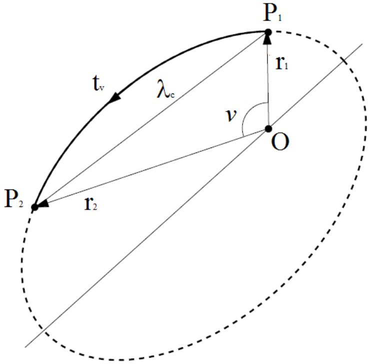
- **Complete** the SDG-code to the Kepler equation for **hyperbolic orbits**:

$$x = e \sinh y - y, \quad e > 1$$



- **Apply** the SDG-code to the **Lambert's problem**: determination of an orbit from two position vectors $(\vec{r_1}, \vec{r_2})$ and the time of flight t_v.

  * In low thrust trajectories



References

- W. NG, Edward, 1979: A general algorithm for the solution of Kepler’s equation for elliptic orbits. Celestial Mechanics.
- Conway, Bruce A, 1986: An improved algorithm due to Laguerre for the solution of Kepler’s equation. Celestial Mechanics.
- Serafin, RA, 1986: Bounds on the solution to Kepler’s equation. Celestial Mechanics.
- Gooding, RH and Odell, AW, 1988: The hyperbolic Kepler equation (and the elliptic equation revisited). Celestial Mechanics.
- Nijenhuis, Albert, 1991: Solving Kepler’s equation with high efficiency and accuracy. Celestial Mechanics and Dynamical Astronomy.
- Colwell, Peter, 1993: Solving Kepler’s equation over three centuries. Richmond, Va: Willmann-Bell.
- Markley, F Landis, 1995: Kepler equation solver. Celestial Mechanics and Dynamical Astronomy.
- Fukushima, Toshio, 1996: A method solving Kepler’s equation without transcendental function evaluations. Celestial Mechanics and Dynamical Astronomy.
- Fukushima, Toshio, 1997: A method solving Kepler’s for hyperbolic case. Celestial Mechanics and Dynamical Astronomy.
- Mortari, Daniele and Elipe, Antonio, 2014: Solving Kepler’s equation using implicit functions. Celestial Mechanics and Dynamical Astronomy.

Vielen Dank für Ihre Aufmerksamkeit