

AN EFFICIENT CODE TO SOLVE THE KEPLER'S EQUATION FOR ELLIPTIC AND HYPERBOLIC ORBITS

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ABSTRACT

The Kepler equation for the elliptical motion, $y - e \sin y = x$, involves a nonlinear function depending on three parameters: the eccentric anomaly $y \equiv E$, the eccentricity e and the mean anomaly $x \equiv M$. For given e and x values the numerical solution of the Kepler equation becomes one of the goals of orbit propagation to provide the position of the object orbiting around a body for some specific time (see references). In this paper, a new approach for solving Kepler equation for elliptical and hyperbolic orbits is developed. This new approach takes advantage of the very good behavior of the Laguerre method [1] when the initial seed is close to the looked for solution and also of the existence of symbolic manipulators which facilitates the obtention of polynomial approximations. The central idea is to provide an initial seed as good as we can to the modified Newton-Raphson method, because when the initial guess is close to the solution, the algorithm is fast, reliable and very stable. To determine a good initial seed the domain of the equation is discretized in several intervals and for each one of these intervals a fifth degree interpolating polynomial is introduced. The six coefficients of the polynomial are obtained by requiring six conditions at both ends of the corresponding interval. Thus the real function and the polynomial have equal values at both ends of the interval. Similarly relations are imposed for the two first derivatives. Consequently, given e and $x \equiv M$, selecting the interval $[x_i, x_{i+1}]$ in such a way that $M \in [x_i, x_{i+1}]$ and using the corresponding polynomial $p_i(x)$, we determine the starter value $y_0 = E_0$. However, the Kepler equation has a singular behavior when M is small and e close to unity (singular corner). In this case, the exact solution of the equation has to be described in a different way to guarantee the enough accuracy to be part of the seed used to start the numerical method. In order to do that, an asymptotic expansion in power of the small parameter $\varepsilon = 1 - e$ is developed. In most of the cases, the seed generated by the Space Dynamics Group at UPM (SDG-code) leads to reach machine error accuracy with the modified Newton-Raphson methods with no iterations or just one iteration. The final algorithm is very stable and reliable. This approach improves the computational time compared with other methods currently in use. The advantage of our approach is its applicability to other problems as for example the Lambert problem for low thrust

trajectories.

Index Terms— Kepler equation, Newton-Raphson method, orbit propagation

1. INTRODUCTION

Over the centuries the resolution of the Kepler equation has been studied for a wide variety of scientists such that in virtually every decade from 1650 to the present there have appeared papers devoted to the Kepler problem and its solution [2]. Although there exists a unique solution of the Kepler equation, there are many methods to describe or approximate it. Basically it depends on the solvers's motivations and the mathematical tools available in the epoch of study. In the last century, with the advent of calculators and computers, there is no impediment to achieving quick solutions of great accuracy. In particular, the symbolic manipulators like Matlab, Maple or Mathematica, are very powerful calculators which are easy to use and have a very intuitive syntax. Besides, they can be compiled to provide the corresponding code in some of the standard programming languages such as C, C++ or Fortran, which are specially adapted for numerical calculus.

In this paper we develop a procedure to solve elliptic and hyperbolic orbits taking advantage of the full potential of the symbolic manipulators. We focus the study on the determination of an appropriate seed to initialize the numerical method for solving the Kepler equation, considering the optimization already tested of the well known Newton-Raphson method.

1.1. Elliptical case

The Kepler equation corresponding to the elliptical motion

$$x = y - e \sin y \quad (1)$$

determines a nonlinear function $y = y(x, e)$ where $y \equiv E$ is the unknown eccentric anomaly, e the eccentricity and $x \equiv M$ the mean anomaly, which is known. For given values of $e \in [0, 1[$ and $M \in [0, 2\pi]$, equation (1) defines univocally a function $y = y(x, e)$. This property can be deduced from the Banach fixed-point theorem. If we consider the equation $g(y) = x + e \sin y$, the function $g(y)$ is a contractive mapping for every value of

x , since its derivative $g'(y) = e \cos y$ satisfies the condition $|g'(y)| < 1$ when $e \in [0, 1[$. In effect, the function $g(y)$ is contractive if there is a nonnegative real number $0 < k < 1$ such that $\forall y_1, y_2 \in [0, 2\pi]$:

$$|g(y_1) - g(y_2)| \leq k|y_1 - y_2|$$

Here, the smallest value for k is called the Lipschitz constant of g . Applying the mean value theorem to function $g(y)$, there exist an intermediate value $y^* \in [y_1, y_2]$ which verifies:

$$g(y_2) - g(y_1) = g'(y^*)(y_2 - y_1)$$

Taking into account that $g'(y) = e \cos y$, we have

$$|g(y_2) - g(y_1)| = |e \cos y^*| |y_2 - y_1| \leq e|y_2 - y_1|$$

that is, the Lipschitz constant of g , k , verifies $k < e$, proving the property. Finally, applying the Banach fixed-point theorem for contractive functions, it can be concluded that there is a fixed point which verifies $y = g(y)$ and this fixed point is the looked for solution of the Kepler equation. Starting from a given value y_0 the sequence $y_{n+1} = g(y_n)$ turns out to be convergent and its limit is the fixed point.

Initially, x and y range in the interval $[0, 2\pi]$. However, the transformation $y = 2\pi - \eta$ and $x = 2\pi - \xi$ convert equation (1) into $\xi = \eta - e \sin \eta$. Therefore, if $x \in]\pi, 2\pi]$ the mapping reduces the problem to the interval $[0, \pi]$. As a consequence, the Kepler equation can be seen as a root finder problem of the equation

$$f(y) \equiv y - e \sin y - x = 0 \quad (2)$$

where $e \in [0, 1[$ and $x \in [0, \pi]$ are known.

1.2. Hyperbolic case

The Kepler equation corresponding to the hyperbolic motion

$$x = e \sinh y - y \quad (3)$$

determines a nonlinear function $y = y(x, e)$ where $y \equiv H$ is the unknown hyperbolic anomaly, e the eccentricity and $x \equiv M_H$, which is known, is the equivalent to the mean anomaly in the elliptic motion. Initially, x and y range in the interval $[-\infty, \infty[$ with $e > 1$. Because the eccentricity e is larger than 1, $y > 0 \Leftrightarrow x > 0$. Besides, the right hand side of (3) is an odd function. That is, for negative values of x the change of variables $x = -\xi$ and $y = -\eta$ convert equation (3) into $\xi = e \sinh \eta - \eta$, where ξ and η are positives. Therefore, we focus the analysis in the resolution of equation (3) for positive values: $(x, y) \in [0, \infty[\times [0, \infty[$.

For given values of $e > 1$ and $M_H \in [0, \infty[$ equation (3) defines univocally a bijective function $y = y(x, e)$ which is defined on \mathbb{R}_+ . This property can be deduced from the Bolzano's theorem and the fact the function $f(y) =$

$e \sinh y - y - x$ is strictly increasing. Indeed, $f(y)$ is a continuous function on the closed interval $[y_1, y_2] = [0, \frac{x}{e-1}]$ such that takes the values of the opposite sign at the extremes

$$f(0) = -x < 0$$

$$f\left(\frac{x}{e-1}\right) = e \sinh\left(\frac{x}{e-1}\right) - \frac{x}{e-1} - x = e \left(\frac{1}{3!} \left(\frac{x}{e-1}\right)^3 + \frac{1}{5!} \left(\frac{x}{e-1}\right)^5 + \dots \right) > 0$$

Applying the Bolzano's theorem to function $f(y)$, there exist an intermediate value $y^* \in [y_1, y_2]$ which verifies $f(y^*) = 0$. Now we can assure $f(y)$ vanishes at least one in the stated interval. However, the derivative $f(y)'$ is always positive

$$\frac{df}{dy} = e \cosh y - 1 \geq e - 1 > 0$$

That is, $\forall y_2 > y_1$ with $y_1, y_2 \in [0, \infty[$, $f(y_2) \geq f(y_1)$ and $f(y)$ is strictly increasing. Consequently, $f(y)$ is zero for only one value of $y \in [y_1, y_2]$, i.e., for given x , $\exists!$ y such that verifies (3).

As a consequence, the Kepler equation can be seen as a root finder problem of the equation

$$f(y) \equiv e \sinh y - y - x = 0 \quad (4)$$

where $e > 1$ and $x \in [0, \infty[$ are known.

1.3. Kepler equation solvers

Several authors have developed different methods to solve the Kepler equation focusing in the accuracy and the computational cost of the algorithm. These aspects depend on how the seed is chosen as well as the method (numerical or analytical) used. In what follows we describe some of the approaches currently used

- The Gooding 1988 approach for hyperbolic case [3]: The starter is based on Lagrange's theorem, where the equation is rewritten as a function of $S = \sinh y$ instead of y . First, the Halley's method is applied to obtain the corrector of S as well as of the function and its derivative. With these values, the Newton-Rhapson method is applied to determine the solution of the equation. It is an iterative method which requires several transcendental function evaluations.
- The Markley 1995 approach for elliptical case [4]: The starter point for y is the root of a cubic equation, whose expression is derived through a Padé approximation for $\sin y$, which is replaced in (2). After that, a fifth-order refinement of the solution is applied only once. It is a no iterative method which requires four transcendental function evaluations.

- The Fukushima 1996 approach for elliptical case [5]: The starter value for y is a trivial upper bound of the solution, π . The approximate solution is determined by transforming the Newton-Raphson method from the continuous x -space to a discretized j -space, where x represents y and j the solution index. The corrected value for y is another approximation of the Newton-Raphson method by approximating the evaluation of (2) and the derivative by their truncated Taylor series around the approximate solution. It is an iterative method which does not require transcendental function evaluation.
- The Fukushima 1997 approach for hyperbolic case [6]: Several intervals of $L=x/e$ are defined to select where the solution should be determine. The different methods and approximations chosen for each interval are going to depend on the expected size solution. First, four cases are selected where the solution is large. For these cases an asymptotic form of the main equation is considered and an approximate solution is found. Next, four other cases are selected when the solution is small. In that case an iterative procedure is applied to solve the approximate forms of Kepler's equation. In the rest case, when the solution interval becomes finite, a discretized Newton method is applied as well as a Newton method where the functions are evaluated by Taylor series expansions. In that case, the starter value is the minimum of some upper bounds of the solution prepared by using the Newton correction formula. This approach, depending on the case, use or not an iterative method, requiring several transcendental function evaluations.
- The Mortari and Elipe 2014 approach for elliptical case [7]: Two ranges of x are defined to select where the solution should be determine. The lower bound for y is derived through two implicit functions, which are non-rational Bézier functions, linear or quadratic, depending on the derivatives of the initial bound values. The upper bound for y is estimated by applying the Newton-Raphson method with the lower bound as the starting value. After that, if the machine error accuracy is not reached, the lower and upper bounds define a new range of searching. It is an iterative method which requires several transcendental function evaluations.

2. THE ITERATIVE METHOD

The method used in our code to solve the Kepler equation is the modified Newton-Raphson method. It is a successive approximation method starting from a seed y_0 , where the root searched is solution of (2) or (4) depending on the case we are. If y_n in one of the terms of the sequence we want to generate, the next term will be $y_{n+1} = y_n + \Delta y_n$, where Δy_n

should verify $f(y_{n+1}) = f(y_n + \Delta y_n) = 0$, with f given by (2) or (4). If we approximate $f(y_{n+1})$ by its second order Taylor expansion about y_n

$$f(y_{n+1}) \approx f(y_n) + f'(y_n)\Delta y_n + \frac{1}{2}f''(y_n)\Delta^2 y_n$$

so that imposing the condition $f(y_{n+1})=0$, we obtain

$$\Delta y_n = \frac{-2f(y_n)}{f'(y_n) \pm \sqrt{|f'^2(y_n) - 2f(y_n)f''(y_n)|}} \quad (5)$$

where we have to select the sign (+) when $f'(y_n)$ is positive; if $f'(y_n) < 0$ the sign (-) must be taken.

The use of absolute value in (5) does not affect the convergence of the algorithm. It is introduced to avoid the algorithm fails during the series generation when the square root is a complex number. The first and second derivatives of (2) are given by the relations

$$\begin{aligned} f'(y_n) &= 1 - e \cos y_n \\ f''(y_n) &= e \sin y_n \end{aligned}$$

while the first and second derivatives of (4) are given by

$$\begin{aligned} f'(y_n) &= e \cosh y_n - 1 \\ f''(y_n) &= e \sinh y_n \end{aligned}$$

Equation (5) summarizes the modified Newton-Raphson algorithm. It should be noticed that this algorithm becomes the classical Newton-Raphson method, in which the function is approximated by its first order Taylor expansion, when $f''(y_n) = 0$. The modified Newton-Raphson is a particular case of the Conway method. Bruce M. Conway, of the University of Illinois, applied a root-finding method of Laguerre (1834-1886) to the solution of Kepler equation (Conway 1986). Although the method is intended for finding the roots of a polynomial, it works equally as well for Kepler equation. The algorithm associated to that method summarizes in the equation

$$\Delta y_n = \frac{-(1+p)f(y_n)}{f'(y_n) \pm \sqrt{|p[pf'^2(y_n) - (1+p)f(y_n)f''(y_n)]|}}$$

with $p = m - 1$ and m is the degree of the polynomial. As we can see, the modified Newton-Raphson method used in this paper is a particular case of the Conway method for $m = 2$.

3. THE SEED VALUE

In order to initialize the iterative method, we require a starting value of the eccentric or hyperbolic anomaly, whose selection should be done carefully to optimize the corresponding method. To do that, we start defining the interval where we are going to work.

3.1. Elliptical case

We start discretizing the eccentric anomaly domain $[0, \pi]$ in twelve intervals of 15° of longitude, where the corresponding $M_i \equiv x_i$ is obtained from equation (1). This way, we defined twelve intervals $[x_i, x_{i+1}]$, $i = 1, \dots, 12$. In each one of these intervals we introduce a fifth degree polynomial $p_i(x)$, $i = 1, \dots, 12$ to interpolate the eccentric anomaly

$$y(x) \approx p_i(x) = a_0^{(i)} + a_1^{(i)}x + a_2^{(i)}x^2 + a_3^{(i)}x^3 + a_4^{(i)}x^4 + a_5^{(i)}x^5$$

in $[x_i, x_{i+1}]$, $i = 1, \dots, 12$.

In order to determine the six coefficients of $p_i(x)$ the following six boundary conditions should be imposed

$$\begin{aligned} p_i(x_i) &= y(x_i) & p_i(x_{i+1}) &= y(x_{i+1}) \\ p_i'(x_i) &= y'(x_i) & p_i'(x_{i+1}) &= y'(x_{i+1}) \\ p_i''(x_i) &= y''(x_i) & p_i''(x_{i+1}) &= y''(x_{i+1}) \end{aligned}$$

where the two first derivatives of y are given by:

$$\begin{aligned} y' &= \frac{dy}{dx} = \frac{1}{1 - e \cos y} \\ y'' &= \frac{d^2y}{dx^2} = \frac{-e \sin y}{(1 - e \cos y)^3} \end{aligned}$$

These polynomials $p_i(x)$, $i = 1, \dots, 12$, can be generated easily with the help of the Maple symbolic manipulator only once; moreover, the Maple manipulator provides the FORTRAN or C code of the polynomials that will be appropriately stored in one module of code.

Consequently, given e and M , we determine the starter value E_0 following the next procedure:

1. First, given e the ends x_i, x_{i+1} of the intervals are calculated applying (1) to the corresponding y_i, y_{i+1} .
2. The interval $[x_i, x_{i+1}]$ should be selected in such a way that $M \in [x_i, x_{i+1}]$. Note that the corresponding values $[y_i, y_{i+1}]$ are known and the seed y_0 satisfies: $y_0 \in [y_i, y_{i+1}]$
3. Use the corresponding polynomial $p_i(x)$ to obtain the seed to be used with the Newton-Raphson algorithm: $y_0 = p_i(M)$

3.2. Hyperbolic case

We start doing a change of variable in (3) such that

$$z = \tanh y \quad \Leftrightarrow \quad y = \frac{1}{2} \ln \frac{1+z}{1-z} \quad (6)$$

which transform the Kepler equation in the relation

$$x = e \frac{z}{\sqrt{1-z^2}} - \frac{1}{2} \ln \frac{1+z}{1-z} \quad (7)$$

but now the variable z ranges in the interval $[0, 1[$ when $y \in [0, \infty[$. Notice that the value $z = 1$ is singular. Equation (7) defines a function $z = z(x, e)$ which will be approximated by fifth degree polynomials as it was explained for the elliptical case. We divide the interval $[0, 1[$ where z ranges in 12 uneven intervals; the lower ends of such intervals are given by:

$$z_i = 0.99 \cdot \left(\frac{i-1}{11} \right)^{1/5}, \quad i = 1, \dots, 12$$

The last interval $z \in [0.99, 1[$, where the value of x could be very large, deserves a different treatment.

- For the first eleven intervals we introduce 11 polynomials of fifth degree $p_i(x)$, $i = 1, \dots, 11$ and in each one of the intervals the function $z(x)$ will be approximated by the corresponding polynomial: $z(x) \approx p_i(x)$, when $x \in [x_i, x_{i+1}]$. As in the elliptical case, the six coefficients of each polynomial are determined by requiring that the function $z(x)$ and the polynomial reach the same values at the ends of the corresponding interval; moreover, the two first derivatives of $z(x)$ and the polynomial must reach the same values at both ends. Now the two first derivatives of z are given by:

$$\begin{aligned} z' &= \frac{dz}{dx} = \frac{(1-z^2)^{\frac{3}{2}}}{e - \sqrt{1-z^2}} \\ z'' &= \frac{d^2z}{dx^2} = \frac{z(2\sqrt{1-z^2} - 3e)(1-z^2)^2}{(e - \sqrt{1-z^2})^3} \end{aligned}$$

- In the last interval $z \in [0.99, 1[$ a recursive algorithm is applied to estimate the seed of z , z_0 . We perform a change of variable in (7) such that $z = \cos \xi$. Because we are in the vicinity of $z = 1$, $\xi \ll 1$ and $x \gg 1$. As a consequence, (7) is given by

$$x = e \cot \xi - \frac{1}{2} \ln(\cot^2 \frac{\xi}{2})$$

being the final equation for the algorithm

$$\xi = h(\xi, e, x) = \arctan \frac{e}{x + \frac{1}{2} \ln(\cot^2 \frac{\xi}{2})} \quad (8)$$

Starting from the value $\xi_0 = \frac{\pi}{2}$ the sequence $\xi_{n+1} = h(\xi_n, e, x)$ turns out to be quickly convergent and its limit ξ_ℓ provides the seed z_0 for z in this last interval: $z_0 = \cos \xi_\ell$.

Consequently, given e and M_H , we determine the starter value z_0 following the next procedure:

1. First, given e the ends x_i, x_{i+1} of the intervals are calculated applying (3) to the corresponding y_i, y_{i+1}

2. The interval $[x_i, x_{i+1}]$ should be selected in such a way that $M_H \in [x_i, x_{i+1}]$. Note that the corresponding values $[z_i, z_{i+1}]$ are known and the seed z_0 satisfies: $z_0 \in [z_i, z_{i+1}]$
3. If the right interval is the last one, use the sequence associated with equation (8); with 4 or 6 iteration we obtain a very good seed $z_0 = \cos \xi_\ell$ to be used with the Newton-Raphson algorithm
4. If the right interval is not the last one, use the corresponding polynomial $p_i(x)$ to obtain the seed $z_0 = p_i(M_H)$
5. The seed z_0 is used to solve the equation (7) by using the modified Newton-Raphson algorithm. The output of this algorithm, $z(x)$ provides, via the change of variable (6), the desired value of the hyperbolic anomaly $y(x)$

4. THE SINGULAR CORNER

4.1. Elliptical case

The expression (2) defines a real function $y = y(x, e)$ which is univocally determined when $e \in [0, 1]$. However, the Kepler equation has a singular behavior in the neighborhood of $e = 1$ and $M = 0$ (singular corner). To describe the solution close to the singular corner we introduce the value $\varepsilon = 1 - e$ assuming that $\varepsilon \ll 1$:

$$y - (1 - \varepsilon) \sin y - x = 0 \quad (9)$$

Our goal is to describe numerically the exact solution y_v of equation (9) with enough accuracy to be part of the seed used to start the Newton-Raphson convergent process. In order to do that, an asymptotic expansion in power of the small parameter ε will be obtained.

In the case $\varepsilon = 0$ the solution of equation (9) is a smooth function $\eta_0(x)$ which is defined in the whole interval $[0, \pi]$. Let us assume that this function is known; in such a case it is possible to obtain an asymptotic solution of equation (9) in the limit $\varepsilon \rightarrow 0$ but for $\varepsilon \neq 0$:

$$y(x) = \eta_0(x) + \varepsilon \eta_1(x) + \varepsilon^2 \eta_2(x) + \dots$$

By introducing this expansion, respectively, in equation (9) and requiring that the resulting series vanish for every order in ε it is possible to obtain the different function $\eta_i(x)$, $i = 1, \dots, n$ in terms of the known function $\eta_0(x)$. With the help of the Maple symbolic manipulator we obtain:

$$y_{as}(x) = \eta_0 - \varepsilon \frac{\sin \eta_0}{1 - \cos \eta_0} + \varepsilon^2 \frac{1}{2} \frac{\sin \eta_0}{1 - \cos \eta_0} + \varepsilon^3 \frac{1}{3} \frac{\cos \eta_0}{\sin \eta_0} \frac{(2 - \cos \eta_0)(1 + \cos \eta_0)}{(1 - \cos \eta_0)^2} \dots \quad (10)$$

In the singular corner η_0 is small and the asymptotic solution (15) involves two small quantities: ε and η_0 . Let us consider a tolerance level ε_{tol} , for example, $\varepsilon_{tol} = 0.001$. When η_0 is clearly larger than ε , the asymptotic solution (10) provides a very accurate result, that is, the difference between the exact value y_v and the asymptotic solution y_{as} is smaller than ε_{tol} : $|y_v - y_{as}| < \varepsilon_{tol}$.

However, for decreasing values of y_0 the asymptotic solution reaches the point (x^*, y^*) where, for the first time, the accuracy fails since the difference $|y_v - y_{as}| \geq \varepsilon_{tol}$. Once the tolerance level ε_{tol} has been fixed, these critical values (x^*, y^*) only depend of the parameter ε . As a consequence, the asymptotic solution (10) cannot be used when $x < x^*$ and we need to look for another solution valid in this zone.

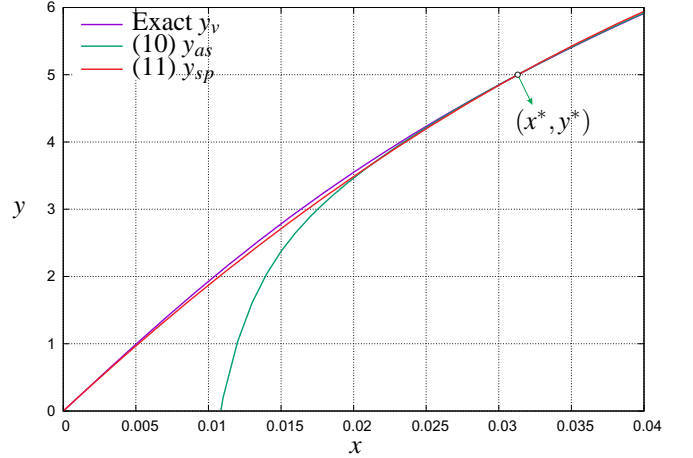


Fig. 1. Exact, asymptotic and special solution for $\varepsilon = 0.005$ in the elliptical case

In this zone, $x < x^*$, we use the following polynomial approximation:

$$y_{sp} = x \left(ax + \frac{1}{\varepsilon} \right), \quad \text{with} \quad a = \frac{y^*}{(x^*)^2} - \frac{1}{\varepsilon x^*} \quad (11)$$

This approximation provides a curve which pass through the origin $(0,0)$, the point (x^*, y^*) and matches the slope at the origin.

Figure 1 shows the exact solution, y_v , the asymptotic solution, y_{as} , and the special solution, y_{sp} for the particular case $\varepsilon = 0.005$ and a tolerance level $\varepsilon_{tol} = 0.001$.

The definition of the point (x^*, y^*) is not precise; however, considering that we are looking for a seed in order to feed the further Newton-Raphson process, the accuracy is enough with tolerance levels of the order of 10^{-3} as we have seen previously.

Summarizing this analysis, to obtain a reliable procedure we have to provide: a reliable and accurate solution for the function $\eta_0(x)$ which is involved in the asymptotic solution y_{as} and the critical point (x^*, y^*) which defines the special solution y_{sp} .

4.1.1. Solution $\eta_0(x)$

The procedure requires the solution $\eta_0(x)$ for small values of x , in order to be in the singular corner. This solution can be obtained using asymptotic expansion in terms of the mean anomaly x :

$$\eta_0(x) = (6x)^{\frac{1}{3}} + \frac{1}{10}x + \frac{1}{1400}(6x)^{\frac{5}{3}} + \frac{1}{4200}(6x)^{\frac{7}{3}} + \dots$$

The number of terms of this expansion can be increased easily and this asymptotic solution provides the value of $y_0(x)$ with greater accuracy than necessary. Note that using an expansion of $\eta_0(x)$ with four terms the differences are of the order of 10^{-6} for small values of x .

4.1.2. Critical values (x^*, y^*)

Once the tolerance level ε_{tol} has been fixed, the critical point (x^*, y^*) is a function of the parameter ε . With the help of the Maple symbolic manipulator is easy to obtain a set of critical pairs (x^*, y^*) for different values of ε and a tolerance of $\varepsilon_{tol} = 5.0 \cdot 10^{-4}$. After that, we use a Least Square fitting procedure to obtain the following functions:

$$x^* = \varepsilon(-86.3921\varepsilon^2 + 9.1074\varepsilon + 0.051632) \quad (12)$$

$$y^* = \sqrt{\varepsilon}(0.99722 + 12.07850\varepsilon - 220.15880\varepsilon^2) \quad (13)$$

which provides the values (x^*, y^*) in terms of $\varepsilon = 1 - e$.

4.1.3. Algorithm in the singular corner

The algorithm used in our code for values of e and M in the singular corner is as follows:

1. We define arbitrarily the singular corner as the region of the plane (e, M) where $e > 0.975$ and $M < 0.05$ rad. We fix $\varepsilon_{tol} = 5.0 \cdot 10^{-4}$.
2. We calculate $\varepsilon = 1 - e$ and if the value of M is larger than the critical value x^* , i.e., $M > x^*(\varepsilon)$ then we calculate the eccentric anomaly using the asymptotic expansion y_{as} given by (10)
3. If the value of M is smaller than the critical value x^* , i.e., $M \leq x^*(\varepsilon)$ then we calculate the eccentric anomaly using the special solution y_{sp} given by (11). In such a case we need to calculate both critical values (x^*, y^*) using equations (12-13)

The eccentric anomaly obtained is the seed y_0 used to feed the Newton-Raphson process.

4.2. Hyperbolic case

The expression (4) defines a real function univocally determined when $e > 1$. However, the Kepler equation has a singular behavior in the neighborhood of $e = 1$ and $M_H = 0$ (singular corner). To describe the solution close to the singular

corner we introduce the value $\varepsilon = e - 1$ assuming that $\varepsilon \ll 1$:

$$y - (1 + \varepsilon) \sinh y + x = 0 \quad (14)$$

Our goal is to describe numerically the exact solution y_v of equation (14) with enough accuracy to be part of the seed used to start the Newton-Raphson convergent process. In order to do that, an asymptotic expansion in power of the small parameter ε will be obtained.

In the case $\varepsilon = 0$ the solution of equation (14) is a smooth function $\eta_0(x)$ which is defined in the whole real line \mathbb{R}_+ . Let us assume that this function is known; in such a case it is possible to obtain an asymptotic solution of equation (14) in the limit $\varepsilon \rightarrow 0$ but for $\varepsilon \neq 0$:

$$y(x) = \eta_0(x) + \varepsilon \eta_1(x) + \varepsilon^2 \eta_2(x) + \dots$$

By introducing this expansion in equation (14) and requiring that the resulting series vanish for every order in ε it is possible to obtain the different functions $\eta_i(x)$, $i = 1, \dots, n$ in terms of the known function $\eta_0(x)$. With the help of the Maple symbolic manipulator we obtain:

$$y_{as}(x) = \eta_0 + \varepsilon \frac{\sinh \eta_0}{1 - \cosh \eta_0} - \varepsilon^2 \frac{1}{2} \frac{\sinh \eta_0}{1 - \cosh \eta_0} + \varepsilon^3 \frac{1}{3} \cosh \eta_0 \sinh \eta_0 \frac{(2 - \cosh \eta_0)(1 + \cosh \eta_0)}{(1 - \cosh \eta_0)^2} \dots \quad (15)$$

In the singular corner η_0 is small and the asymptotic solution (15) involves two small quantities: ε and η_0 . Let us consider a tolerance level ε_{tol} , for example, $\varepsilon_{tol} = 0.0015$. When η_0 is clearly larger than ε , the asymptotic solution (15) is an excellent approximation of the true solution $y_v(x)$, that is, the difference between the exact value y_v and the asymptotic solution y_{as} is smaller than ε_{tol} : $|y_v - y_{as}| < \varepsilon_{tol}$.

However, for decreasing values η_0 the asymptotic solution behaves worse and it reaches the point (x^*, y^*) where, for the first time, the accuracy fails since the difference $|y_v - y_{as}| \geq \varepsilon_{tol}$. Once the tolerance level ε_{tol} has been fixed, these critical values (x^*, y^*) only depend of the parameter ε . As a consequence, the asymptotic solution (15) cannot be used when $x < x^*$ and we need to look for another solution valid in this zone.

In this zone, $x < x^*$, we use the following polynomial approximation:

$$y_{sp} = x(ax + \frac{1}{\varepsilon}), \quad \text{with} \quad a = \frac{y^*}{(x^*)^2} - \frac{1}{\varepsilon x^*} \quad (16)$$

This approximation provides a curve which pass through the origin $(0, 0)$, the point (x^*, y^*) and matches the slope at the origin.

Figure 2 shows the exact solution, y_v , the asymptotic solution, y_{as} , and the critical values, (x^*, y^*) for the particular cases $\varepsilon = 0.001, 0.01, 0.03, 0.05$ and a tolerance level $\varepsilon_{tol} =$

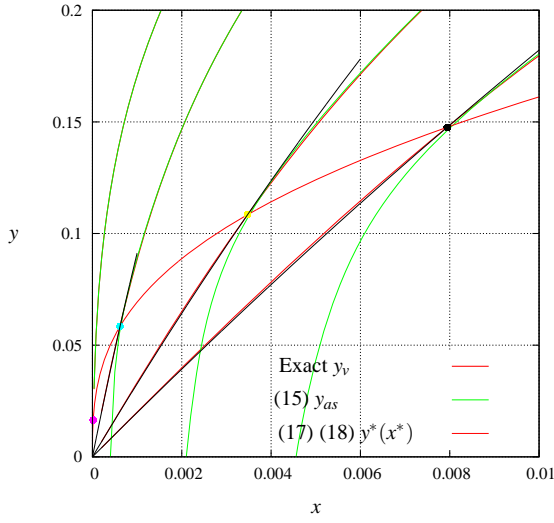


Fig. 2. Exact, asymptotic and critical values for $\varepsilon = 0.001, 0.01, 0.03, 0.05$ in the hyperbolic case

0.0015. We can see the progressive separation of the functions y_v and y_{as} for decreasing values of x .

Summarizing this analysis, to obtain a reliable procedure we have to provide: a reliable and accurate solution for the function $\eta_0(x)$ which is involved in the asymptotic solution y_{as} and the critical point (x^*, y^*) which defines the special solution y_{sp} .

4.2.1. Solution $\eta_0(x)$

The procedure requires the solution $\eta_0(x)$ for small values of x , in order to be in the singular corner. This solution can be obtained using asymptotic expansion in terms of the mean anomaly x :

$$\eta_0(x) = 1.817121x^{\frac{1}{3}} - \frac{1}{10}x + 0.0141511x^{\frac{5}{3}} - 0.0025959x^{\frac{7}{3}} + 0.0005384972x^3 + \dots$$

For $x = 2$, for example, by using an expansion of $\eta_0(x)$ with six terms the relative error is of the order of 10^{-7} . But for small values of x the behavior of the asymptotic solution is better and by using an expansion with three terms the relative error is of the order of 10^{-6} .

4.2.2. Critical values (x^*, y^*)

Once the tolerance level ε_{tol} has been fixed, the critical point (x^*, y^*) is a function of the parameter ε . With the help of the Maple symbolic manipulator is easy to obtain a set of critical pairs (x^*, y^*) for different values of ε and a tolerance of $\varepsilon_{tol} = 1.5 \cdot 10^{-3}$. After that, we use a Least Square fitting procedure

to obtain the following functions:

$$x^* = 0.023988\varepsilon + 4.300478\varepsilon^2 - 62.308284\varepsilon^3 + 869.102230\varepsilon^4 - 6174.838990\varepsilon^5 + 1.7777.158551\varepsilon^6 \quad (17)$$

$$y^* = \sqrt{\varepsilon}(0.549826 + 3.685319\varepsilon - 53.136123\varepsilon^2 + 584.308539\varepsilon^3 - 2254.963856\varepsilon^4) \quad (18)$$

4.2.3. Algorithm in the singular corner

The algorithm used in our code for values of e and M_H in the singular corner is as follows:

1. We define arbitrarily the singular corner as the region of the plane (e, M_H) where $e < 1.1$ and $M_H < 0.1$ rad. We fix $\varepsilon_{tol} = 1.5 \cdot 10^{-3}$.
2. We calculate $\varepsilon = e - 1$ and if the value of M is larger than the critical value x^* , i.e., $M_H > x^*(\varepsilon)$ then we calculate the hyperbolic anomaly using the asymptotic expansion y_{as} given by (15)
3. If the value of M_H is smaller than the critical value x^* , i.e., $M_H \leq x^*(\varepsilon)$ then we calculate the hyperbolic anomaly using the special solution y_{sp} given by (16). In such a case we need to calculate both critical values (x^*, y^*) using equations (17-18)

5. ANALYSIS OF RESULTS

5.1. Elliptical case

We performed an exhaustive numerical analysis of the algorithm that we propose in these pages. In our calculations, instead of use 12 approximating polynomials we use 23. After check the behavior of the algorithm by using several number of polynomials we came to the conclusion that: 1) the differences are no important when using 12, 16, 18 or 23 polynomials, and 2) by using 23 the algorithm behavior is slightly smoother from a global point of view. All the calculations have been carried out in a workstation with Intel(R) Xeon(R) ES-2620 v2 2.10 GHz microprocessor in a Windows 8.1 64 bits operative system and with the same Intel C/C++ compiler.

The Kepler equation for the elliptic case has been solved with the algorithm proposed in this paper by using five different iteration algorithms:

- The SDG-code with the modified Newton-Raphson (MNR) method
- The SDG-code with the Conway method
- The SDG-code with the classical Newton-Raphson (CNR) method

Scheme	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i \geq 4$
SDG-MNR	0.66	99.33	0.0052	0	0
SDG-Conway	0.66	99.28	0.057	0	0
SDG-CNR	0.66	95.02	4.3	0.0093	0
Fukushima	0.05	0.00	0.35	25.75	73.85
Mortari	0.1	0.05	57.20	44.32	0.33

Table 1. Number of iterations (percentage) obtained with the different Newton-Raphson algorithms checked

- The method described in Fukushima 1996 [5]
- The method described in Mortari and Elipe 2014 [7]

In each case, the Kepler equation has been solved $\approx 4 \cdot 10^6$ times, using quadruple precision and a tolerance equal to the zero of the machine: $\epsilon_{tol} = 2.22 \cdot 10^{-16}$. It should be noticed that when working with tolerances close to the zero of the machine, any numerical procedure based on consecutive approximations is affected by artificial numerical chaos. By using quadruple precision it is possible to escape from such a numerical chaos. In each run we count the number of iterations needed to reach a solution with a residual $\rho = |y - e \sin y - x|$ lower than the tolerance ϵ_{tol} .

Comparing the number of iterations (see table 1) and the time computing between the different methods, we see that:

- Applying the SDG-code and the modified Newton-Raphson (MNR) method we see that in the 99.33 % of cases we reach the solution with only 1 iteration; in the 0.66 % no iteration is necessary and in very small number of cases located in the singular corner —the 0.0052 %— two iteration are required. These results justify the option of work with quadruple precision which does not slow the calculations due to the very small number of iterations required. The CPU time invested in the $\approx 4 \cdot 10^6$ of times that we solved the Kepler equation was 68.4 seconds and the averaged number of iterations was 0.99.
- Applying the SDG-code and the Conway method we see that in the 99.28 % of cases we reach the solution with only 1 iteration; in the 0.66 % no iteration is necessary and in small number of cases located in the singular corner —the 0.057 %— two iteration are required. The CPU time invested was 70.4 seconds and the averaged number of interaction was 0.99.
- Applying the SDG-code and the classical Newton-Raphson (CNR) method we see that in the 95.02 % of cases we reach the solution with only 1 iteration; in the 0.66 % no iteration is necessary and in small number of cases located close to the singular corner —the 4,3 %— two iteration are required; finally in a very small number of cases —0.0093 %— three interactions

are needed. The CPU time invested was 62.1 seconds and the averaged number of interaction was 1.037.

- Applying the code described in Fukushima code [5] we see that in the percentage of cases with 0, 1 or 2 iterations turns out to be about the 0.40 %. In the 25.75 % of cases the number of iteration is three and in the 73.85 % of cases 4 or more iterations are needed. The CPU time invested was 328 seconds and the averaged number of interaction was 4.030. Unfortunately even though the CPU time invested in each interaction is lower, the high number of iterations jeopardize the speed of calculations.
- Applying the code described in Mortari [7] we see that in the 57.20 % of cases 2 iterations are needed and in the 42.32 % of cases three iterations should be performed. The percentage of cases with 1 or 2 iterations turns out too be about the 0.15 %. Only in a 0.33 % the number of iteration is greater than three. The CPU time invested was 101.4 seconds and the averaged number of interaction was 2.427.

5.1.1. Accuracy analysis

In this section we focus the analysis in the SDG-code in which the iteration scheme is provided by the MNR algorithm. In the SDG-code the iteration ends when the residual is lower than the zero of the machine, that is, when $\rho = |y - e \sin y - x| \leq \epsilon_{tol} = 2.22 \cdot 10^{-16}$.

Let us consider the true solution $y_v(x, e)$ for given values of e and the mean anomaly $M = x$. Let $y_c(x, e)$ the numerical solution provided by the SDG-code for these particular values. Obviously, due to truncation and round-off errors it is possible to write

$$y_c = y_v + \epsilon_{abs}$$

where ϵ_{abs} is the total absolute error associated with the numerical solution $y_c(x, e)$. This absolute error is closely related with the residual ρ . In effect, for the numerical solution $y_c(x, e)$ the residual is given by:

$$\rho = |y_v + \epsilon_{abs} - e \sin(y_v + \epsilon_{abs}) - x|$$

Taking into account that the true solution verifies

$$y_v - e \sin y_v - x = 0$$

the residual takes the form

$$\rho = |\epsilon_{abs}| |1 - \cos y_v|$$

As a consequence the total absolute error is given by:

$$|\epsilon_{abs}| = \frac{\rho}{|1 - \cos y_v|}$$

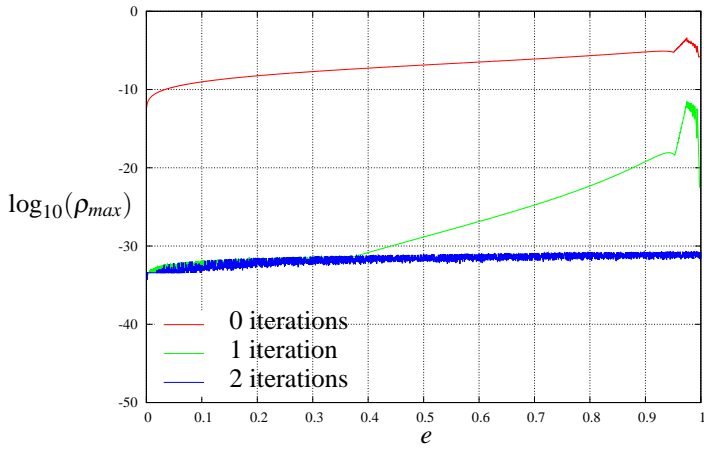


Fig. 3. The maximum residual ρ_{max} for different iterations versus the eccentricity e

Basically, it depends on the residual ρ , just like the total relative error

$$|\epsilon_{rel}| = \frac{|y_c - y_v|}{|y_v|} = \frac{\rho}{|y_v(1 - \cos y_v)|}$$

Let us fix the value of the eccentricity e ; then we scan the whole interval $M \in [0, \pi]$ calculating the residual ρ after zero iteration (the residual provided by the starting seed), after one iteration, two iterations and so on. Let ρ_{max} the maximum residual that we found in the scanning of the whole interval $M \in [0, \pi]$ for each iteration number. These maximum residuals can be associated with the value of the eccentricity e . If we plot the values of such maximum residual we have a very good idea of the accuracy involved in our calculations. Notice that these residuals mark the worst accuracy of the numerical solution $y_c(x, e)$ provided by the SDG-code; in fact, for each value of e there are hundred of values of M where the accuracy is much better than the indicated by ρ_{max}

Figure 3 shows the decimal logarithm of the residual vs. the eccentricity e . The initial seed provides a residual which, in the worst cases, ranges from $10^{-7/2}$ to 10^{-12} . Since our calculations have been carried out in quadruple precision, the residual obtained after one iteration practically saturates the capacity of the machine when $e < 0.4$ (approximately). For $e > 0.4$ the residual grows and reaches a maximum in the singular corner; however, it decreases again when the asymptotic expansion used in that boundary layer comes into play. In any case, after two iterations, the residual reaches—in the whole interval $e \in [0, 1]$ —the minimum values compatible with the quadruple precision used in this test. The figure 3 does not show the results obtained with three or four iterations because cannot be distinguished from the results corresponding to only two iterations. After two iterations the maximum accuracy is obtained and to increase the number of iteration does not improve the quality of the numerical solution. We

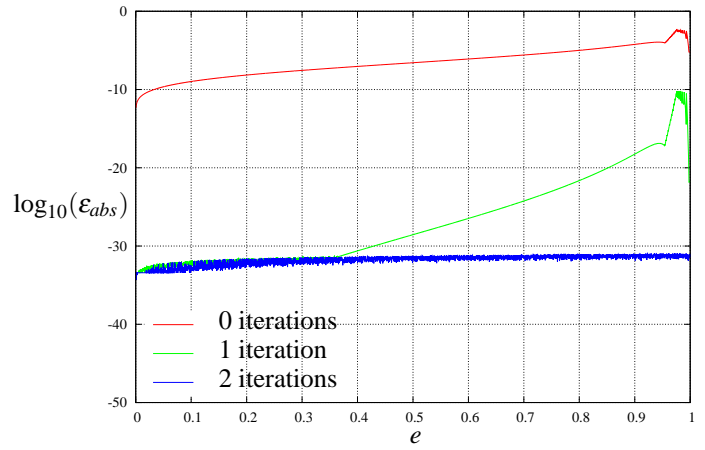


Fig. 4. The maximum total absolute error ϵ_{abs} for different iterations versus the eccentricity e

found a similar situation if instead of the residual we plot the maximum total absolute error ϵ_{abs} . Figure 4 shows the decimal logarithm of ϵ_{abs} vs. e . Practically, the same comments apply to this figure.

5.2. Hyperbolic case

We performed an exhaustive numerical analysis of the algorithm for the hyperbolic case. All the calculations have been carried out in a workstation with Intel(R) Xeon(R) ES-2620 v2 2.10 GHz microprocessor in a Windows 8.1 64 bits operative system and with the same Intel C/C++ compiler.

The Kepler equation has been solved $\approx 16 \cdot 10^6$ times, using quadruple precision and a tolerance equal to the zero of the machine: $\epsilon_{tol} = 2.22 \cdot 10^{-16}$. In each run we count the number of iterations needed to reach a solution with a residual $\rho = e \sinh y - y - x$ lower than the tolerance ϵ_{tol} . The eccentricity ranges in the interval $e \in]1, 5]$ and the values of $M_H = x$ range in the interval $M_H \in [0, 20]$ (rad).

Applying the SDG-code and the modified Newton-Raphson (MNR) iteration scheme, we notice that in the 59.09 % of cases we reach the solution with only 1 iteration; in the 40.88 % of cases two iteration are required. In a very small number of cases—0.00072 %—three iterations are required. Just zero iterations are presented in the 0.026 % of cases. In summary, in the 99.996 % of cases only 0, 1 or 2 iterations are required. These results justify the option of work with quadruple precision which does not slow the calculations due to the very small number of iterations required. The CPU time invested in the $\approx 16 \cdot 10^6$ of times that we solved the Kepler equation was 394 seconds and the averaged number of iterations 1.408.

6. CONCLUSIONS

Several conclusions can be drawn from the previous analysis:

1. An efficient code has been developed to solve the elliptic Kepler equation and preliminary results are provided for the hyperbolic Kepler equation.
2. When the starting seed of the Newton-Raphson algorithm is very good, convergence is always assured as confirmed the thoroughly analysis carried out in our computers.
3. The stability and reliability of our scheme combined with the Newton-Raphson algorithm in its different versions has been assessed.
 - For the elliptical case, the classical Newton-Raphson provides the solution in the 95.02 % of cases with only one iteration. This is remarkable result that we do not found in other algorithms used in the Astrodynamics community. Improving the algorithm by using the Conway or the modified Newton-Raphson permits to obtain the right solution with only one iteration in the 99.3 % of cases. Besides, when quadruple precision is used in numerical simulations, the absolute error of the numerical solution provided by the SDG-code is clearly under the 10^{-20} after one iteration, practically in the whole interval $e \in [0, 1]$ except in the region where $e > 0.877$ (approximately). In any case, after two iterations the error is always lower than 10^{-30} , including the region $e > 0.877$.
 - For the hyperbolic case, the modified Newton-Raphson provides the solution in the 59.09 % of cases with only one iteration and in the 40.88 % of cases with two iterations.
4. The low number of iterations permits to use quadruple precision if you like, because the speed of the calculations is not jeopardized. Therefore, we obtain the benefit of a greater accuracy with a minimal cost.
5. The global algorithm solves successfully the solution of the Kepler equation in the singular corner, $M \ll 1$ and $e \approx 1$. The asymptotic expansions used to generate the initial seed assure the reliability and convergence of the Newton-Raphson iterative scheme. This is another remarkable results since the convergence in this special region not always is assured by some of the algorithms usually considered in the literature.

7. REFERENCES

- [1] Bruce A Conway, "An improved algorithm due to Laguerre for the solution of kepler's equation," *Celestial mechanics*, vol. 39, no. 2, pp. 199–211, 1986.
- [2] Peter Colwell, "Solving kepler's equation over three centuries," *Richmond, Va.: Willmann-Bell, 1993.*, vol. 1, 1993.
- [3] RH Gooding and AW Odell, "The hyperbolic kepler equation (and the elliptic equation revisited)," *Celestial mechanics*, vol. 44, no. 3, pp. 267–282, 1988.
- [4] F Landis Markley, "Kepler equation solver," *Celestial Mechanics and Dynamical Astronomy*, vol. 63, no. 1, pp. 101–111, 1995.
- [5] Toshio Fukushima, "A method solving kepler's equation without transcendental function evaluations," *Celestial Mechanics and Dynamical Astronomy*, vol. 66, no. 3, pp. 309–319, 1996.
- [6] Toshio Fukushima, "A method solving kepler's equation for hyperbolic case," *Celestial Mechanics and Dynamical Astronomy*, vol. 68, no. 2, pp. 121–137, 1997.
- [7] Daniele Mortari and Antonio Elife, "Solving kepler's equation using implicit functions," *Celestial Mechanics and Dynamical Astronomy*, vol. 118, no. 1, pp. 1–11, 2014.