MULTIPLE REVOLUTION LAMBERT´S TARGETING PROBLEM: AN ANALYTICAL APPROXIMATION

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ABSTRACT

Solving a Lambert's Targeting Problem (LTP) consists of obtaining the minimum required single-impulse delta-V to modify the trajectory of a spacecraft in order to transfer it to a selected orbital position in a fixed time of flight. We present an approximate analytical method for rapidly solving a generic LTP with limited loss of accuracy. The solution is built on an optimum single-impulse time-free transfer with a D-matrix phasing correction. Because it exhibits good accuracy (both in terms of delta-V and time of flight) near locally optimum transfer conditions the method is useful for rapidly obtaining low delta-V solutions for interplanetary trajectory optimization. In addition, the method can be employed to provide a first guess solution for enhancing the convergence speed of an accurate numerical Lambert solver.

Index Terms— Lambert's Problem, D Matrix, Quartic Equation, Targeting

1. INTRODUCTION

Lambert's problem (LP), i.e. the determination of the finite set of orbits linking two position vectors in a two-body gravitational field with a specified transfer time, is of fundamental importance in orbital mechanics and has been studied for more than two centuries. If one accounts for multiple revolutions and both prograde and retrograde transfers Lambert's problem provides $2\left(N_{ub}^{D}+N_{ub}^{R}+1\right)$ solutions where N_{ub}^{D},N_{ub}^{R} are, respectively, the maximum number of revolutions that a direct (D) and retrograde (R) orbits can have before reaching the target location[1]. That means that if long-duration transfer arcs are sought the problem becomes more difficult to solve, the longer so the transfer time considered.

The solutions of a Lambert's problem are to be obtained numerically and are essential for various tasks in astrodynamics, including the design of interplanetary trajectories. Most efforts in the literature have been devoted to the improvement of the convergence speed of its numerical solution using different choices for the iteration variable and ways to improve its initial guess. A quite detailed historical review on the different contributions to the method can be found in the introduction of reference[1].

When focusing on the design and optimization of space trajectories one encounters a subclass of Lambert's Problem referred to as Lambert's Targeting Problem (LTP). It consists of obtaining the minimum required single-impulse delta-V to modify the trajectory of a spacecraft in order to transfer it to a selected orbital position in a fixed time of flight. If the solutions of a Lambert's Problem have been obtained the LTP is solved right away by picking the one providing minimum delta-V magnitude within the maximum number of revolutions physically allowed by the transfer time constraint. The reason for distinguishing between LP and LTP is mostly related to the method proposed in the present work, which is concerned with the solution of an LTP but not the general solution of an LP.

Solving an LTP with the highest possible computational efficiency is key to the design of optimized interplanetary trajectories, and, in particular, the ones including multiple gravity assists and deep space maneuvers (DSMs). For these types of problems, the complete sampling (grid search) of the multidimensional solution space implies the solution of a number of LTPs often exceeding the capability of even the most advanced computational means available today. One example is the design and optimization of multiple gravity assist trajectories with multiple DSMs like the Messenger trajectory to Mercury, which employed seven trajectory arcs with five deterministic DSM with delta-V larger than 70 m/s[2]. The sheer number of LTPs to be solved in this case makes the problem computationally intractable if a brute-force, grid-sampling approach is to be followed (see Conway's book[3], page 202). Another relevant application for a high-efficiency LTP solver is the design of "inter-satellitary" trajectories at the giant planets where multi-revolutions transfers are common and the number of possible flyby sequences make the design and optimization task "computationally gigantic"[4]. It is clear that even a modest increase of efficiency in an LTP solution algorithm would be extremely beneficial when facing these computational challenges. In addition, any intrinsic pruning capability that the algorithm could offer in order to discard unintresting transfer arcs a priory is highly welcome.

Recent advances towards a highly efficient LTP solver have been made by these authors[5], who proposed an approximate analytical solution of the LTP based on Battin's optimum single-impulse transfer solution and a linear phasing correction through the use of a novel Keplerian error state transition matrix[6],[7]. The solution provides a useful tool for preliminary trajectory design when high accuracy is not a must as long as the main features of the solution are preserved.

We review the steps leading to the derivation of the analytical LTP solver providing additional formulas and examples not previously described.

2. ANALYTICAL LTP SOLVER

Let us consider the Lambert's Targeting Problem (LTP) of a spacecraft departing from a point $\mathbf{r}_1$ with initial velocity $\mathbf{v}_0$ and arriving at a point $\mathbf{r}_2$ in a fixed time of flight Δt_L while moving in a Keplerian gravitational field of gravitational parameter μ. The required delta-V targeting impulse $\Delta \mathbf{v}_L$ can be broken down into an optimum time-free impulsive delta-V ($\Delta \mathbf{v}_B$) and a phsing correction term ($\Delta \mathbf{v}_C$):

$$\Delta \mathbf{v}_L = \Delta \mathbf{v}_B + \Delta \mathbf{v}_C. \tag{1}$$

Time-free delta-V

The optimum time-free impulsive delta-V ($\Delta \mathbf{v}_B$) can be obtained analytically following the method proposed by Battin, which reduces to the solution of the quartic equation[8]:

$$x^4 - Px^3 + Qx - 1 = 0 \tag{2}$$

where $x > 0$ is the square of the ratio between the skew-radial and skew-chordal components of the optimum departure velocity

$$x^2 = \frac{v_{\rho 1}}{v_{c1}}, \tag{3}$$

while:

$$P = \sqrt{\frac{2 r_1 r_2}{\mu c}} \cos\frac{\Delta\theta}{2} \left(\mathbf{v}_{0\pi} \cdot \mathbf{u}_{r1}\right),$$

$$Q = \sqrt{\frac{2 r_1 r_2}{\mu c}} \cos\frac{\Delta\theta}{2} \left(\mathbf{v}_{0\pi} \cdot \mathbf{u}_{c}\right),$$

with $r_1, r_2, \Delta\theta, c$ indicating, respectively, the magnitudes of the departure and arrival location, the angular width of the transfer arc and its chordal length, while $\mathbf{v}_{0\pi}, \mathbf{u}_{r1}, \mathbf{u}_c$ are, respectively, the transfer plane component of the spacecraft initial velocity $\mathbf{v}_0$, the radial unit vector at departure and the chordal unit vector from $\mathbf{r}_1$ to $\mathbf{r}_2$.

Eq. (2) is a quartic algebraic equation that can be solved analytically using different methods (e.g. by Lagrange resolvent) or numerically (e.g. with Newton's or Halley's method). Once solved, the optimal chordal and radial components of $\mathbf{v}_1$ are determined using Eq. (3) and the following relation:

$$\frac{1}{v_{c1}} = \sqrt{\frac{2 r_1 r_2}{\mu c}} x \cos\frac{\Delta\theta}{2}, \tag{4}$$

Having determined the optimum $\mathbf{v}_1$ the total single-impulse transfer velocity vector is obtained as:

$$\Delta \mathbf{v}_B = \mathbf{v}_1 - \mathbf{v}_0, \tag{5}$$

while the angular momentum and eccentricity vector of the transfer orbit are obtained as:

$$\mathbf{h} = h\mathbf{u}_h = \mathbf{r}_1 \times \mathbf{v}_1$$

$$\mathbf{e} = e\mathbf{u}_e = \frac{\mathbf{v}_1 \times \mathbf{h}}{\mu} - \mathbf{u}_{r1}.$$

The true and eccentric anomalies at $\mathbf{r}_1$ and $\mathbf{r}_2$ can now be determined from the relations:

$$\cos\theta_{1,2} = \mathbf{u}_{r1,2} \cdot \mathbf{u}_e; \qquad \sin\theta_{1,2} = \left(\mathbf{u}_e \times \mathbf{u}_{r1,2}\right) \cdot \mathbf{u}_h,$$

$$\cos E_{1,2} = \frac{e - \cos\theta_{1,2}}{1 + e\cos\theta_{1,2}}; \qquad \sin E_{1,2} = \frac{\sqrt{1-e^2}\sin\theta_{1,2}}{1 + e\cos\theta_{1,2}}.$$

The semi-major axis and mean motion are readily computed as:

$$a = \frac{r_1}{1 - e\cos E_1}, \qquad n = \sqrt{\frac{\mu}{a^3}}$$

Finally the transfer time satisfies Kepler's equation:

$$\Delta t_B = \frac{E_2 - E_1 - e\left(\sin E_2 - \sin E_1\right)}{n}.$$

For hyperbolic orbits ($e > 1$), all preceding relations are still valid after considering the relation between hyperbolic and elliptic eccentric anomaly:

$$H = iE. \tag{6}$$

D matrix targeting

Well-designed transfer arcs are characterized by high phasing efficiencies or, equivalently, small magnitude velocity corrections $\Delta \mathbf{v}_C$. When this is the case one can obtain a sufficiently accurate estimation of $\Delta \mathbf{v}_C$ starting from the optimum single-impulse transfer trajectory. The sought maneuver correction will be the one adjusting the trajectory phasing without changing the orbit radius at the arrival true anomaly θ_2.

The phasing correction to be applied is best estimated as:

$$\delta t = \Delta t_L - \Delta t_B - N\, T_B, \tag{7}$$

where N is the total number of full revolutions of the closest optimum single-impulse transfer:

$$N = \left[\frac{\Delta t_L - \Delta t_B}{T_B} \right],$$

with [] denoting the *nearest integer* function.

The velocity correction $\Delta \mathbf{v}_C$ associated with δt can be obtained from the linear relation linking the radial position shift (δr) and time delay (δt) of a Keplerian orbit at true anomaly θ to the applied delta-V maneuver along the radial ($\delta v_{r,1}$) and transversal direction ($\delta v_{\theta,1}$) at true anomaly θ_1 [6],[7]:

$$\left[\begin{array}{c} \delta r\,(\theta) \\ \delta t\,(\theta) \end{array} \right] \approx \mathbf{D}\,(\theta, \theta_1) \left[\begin{array}{c} \delta v_{r,1} \\ \delta v_{\theta,1} \end{array} \right]. \tag{8}$$

The D matrix is essentially a Keplerian *error state transition matrix* where time (t) is a state variable together with the radial distance (r) and the radial and transversal velocity (v_r, v_θ) while the independent variable is the angular position θ on the reference orbit (conveniently measured from the reference orbit eccentricity vector).

The D matrix terms were first obtained in reference[6] and are here rewritten in a more compact way in terms of angular momentum, radial distance and classical orbital elements of the reference orbit:

$$\mathbf{D}_{1,1} = \frac{r^2}{h} \sin\left(\theta - \theta_1\right),$$

$$\mathbf{D}_{1,2} = \frac{r^2 r_1}{h} \times \frac{2 - 2\cos\left(\theta - \theta_1\right) - e \sin\theta_1 \sin\left(\theta - \theta_1\right)}{a\left(1 - e^2\right)}.$$

$$\begin{aligned} \mathbf{D}_{2,1} &= \frac{a^4\left(1\text{-}e^2\right)}{2 r_1 h^2} \left[-\left(4\Delta_{CE} - e\Delta_{\mathrm{C2E}}\right)\left(\cos E_1 - e\right) \right. \\ &+ \left. \left(6e\Delta_E - 4\left(1 + e^2\right)\Delta_{SE} + e\Delta_{\mathrm{S2E}}\right)\sin E_1 \right], \end{aligned}$$

$$\begin{aligned} \mathbf{D}_{2,2} &= \frac{a^4\sqrt{1 - e^2}}{4 r_1 h^2} \left[12\left(1 - e^2\right)\Delta_E - 3e^2\Delta_{\mathrm{S2E}} \right. \\ & +6e^3\Delta_{SE} + \left(2\left(2 - e^2\right)\sin E_1 - e\sin 2E_1\right) \times \\ & \left(4\Delta_{CE} - e\Delta_{C2E}\right) + \left(4\cos E_1 - e\cos 2E_1\right) \times \\ & \left. \left(e\Delta_{S2E} - 2\Delta_{SE}\left(2 - e^2\right)\right) \right], \end{aligned}$$

where:

$$\begin{aligned} \Delta_E &= E - E_1 \\ \Delta_{SE} &= \sin E - \sin E_1 \\ \Delta_{CE} &= \cos E - \cos E_1 \\ \Delta_{S2E} &= \sin 2E - \sin 2E_1 \\ \Delta_{C2E} &= \cos 2E - \cos 2E_1. \end{aligned}$$

The necessary targeting delta-V conditions are obtained by inversion of Eq.(8) after setting zero radial error et encounters and the required targeting time delay in Eq.(7). In this manner:

$$\left[\begin{array}{c} \delta v_{r,1} \\ \delta v_{\theta,1} \end{array} \right] = \mathbf{D}^* \left[\begin{array}{c} 0 \\ \delta t \end{array} \right].$$

where we have introduced the "guidance matrix" $\mathbf{D}^* = \mathbf{D}^{-1}$. Finally, the sought velocity correction yields:

$$\Delta \mathbf{v}_C \approx \delta v_{r,1}\mathbf{u}_{r1} + \delta v_{\theta,1}\mathbf{u}_\theta. \tag{9}$$

Eq(9), together with (5), provide the approximate solution of the LTP (Eq.(1)).

Performance of the method

The performance of the method, in terms of accuracy and speed, has been tested by producing pork-chop plots of direct transfer trajectories to Mars and to the asteroid 65803 Didymos[9]. In all cases, contour plots were produced for both the departure energy (C_3) and arrival excess velocity at the target celestial body (ΔV). The plots were obtained by a 1000×1000 discretization of the departure epoch × time of flight coordinates leading to a one million Lambert solver calls. The analytical formulas derived in this article were coded in fortran using a GNU gfortran compiler (version 5.3.1) in simple precision with an Intel Core processor i7-4790@3.6GHz. The use of simple precision fits well within the scope of a preliminary optimization tool and its future GPU implementation.

In order to assess the accuracy of the method and its computational speed three very fast Lambert solver algorithms were considered: the classical Gooding's method[10], the recently published Izzo's method[11] and the Arora-Russell method[1]. The double-precision fortran versions of the two former methods were kindly made available by Jacob

Table 1. Simplified ephemerides for the considered celestial bodies

	Earth	Mars	65803 Didymos
reference epoch (MJD)	58849	58849	57200
semi-major axis (AU)	1.00	1.52	1.64
eccentricity	0.0167	0.0934	0.384
inclination (deg)	0.00280	1.85	3.41
argument of pericenter (deg)	287	285	319
longitude of ascending node (deg)	176	49.5	73.2
mean anomaly (deg)	357	247	190

Williams through the Fortran Astrodynamics Toolkit github repository[12]. A double-precision fortran code for the Arora-Russell method was kindly provided to the authors by Nitin Arora and Ryan Russell.

Simplified ephemerides, reported in Table 1, were used in all cases in order to ease the reproducibility of the results.

It is important to underline that in order to obtain sufficiently accurate approximations the analytical LTP solver needs to be run starting from two different optimum single impulse problems. For the departure C_3 estimation problem the targeting is clearly done over an optimum single impulse problem of a trajectory starting from Earth. Conversely, for the arrival excess velocity estimation problem a reversed trajectory starting from the target body and headed towards the Earth is employed (meaning that the sign of all velocity vector components has to be switched).

Mars trajectories

Figures 1 and 2 show the analytically and numerically computed pork-chop plots for an Earth-Mars transfer in the years 2025-2030. Contour plots are given for both departure energy (fig. 1) and arrival excess velocity at Mars (2). The quality of the analytical pork-chop plot is remarkable and all main features appear to be well reproduced. In addition the delta-V and launch date error for all main local optimum solutions are negligible.

As for the computing time the method, provides a reduction of a factor of 4.3, 5.1, 10.0 with respect to the method of Arora-Russell, Izzo and Gooding, respectively. The maximum number of revolutions for the previous methods (which mildly affects Arora-Russell and Izzo's method but does affect Gooding's one considerably) was set to 2 in this case.

Didymos trajectories

Figures 3 and 4 show the analytically and numerically computed pork-chop plots for a transfer to the binary asteroid 65803 Didymos in the years 2019-2022. Contour plots are given for both departure energy (fig. 3) and arrival excess velocity (4). Similarly to the previous case the structure of the pork-chop plots is well reproduced by the analytical method

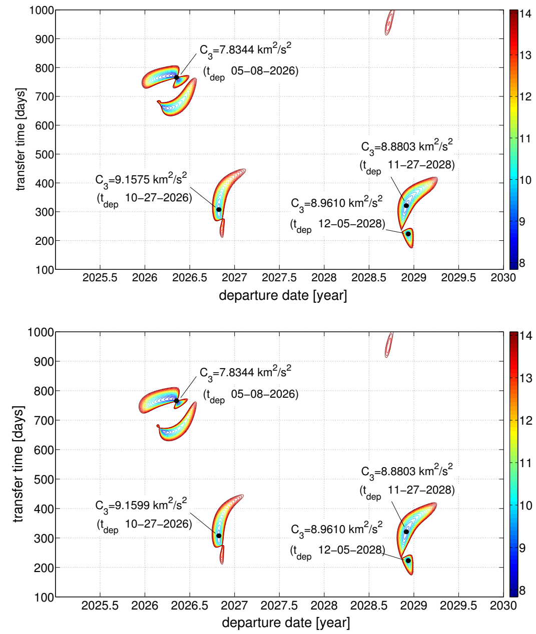

Fig. 1. Numerical (top) and analytical (bottom) pork-chop plot for the injection energy (C_3) of an Earth-Mars transfer.

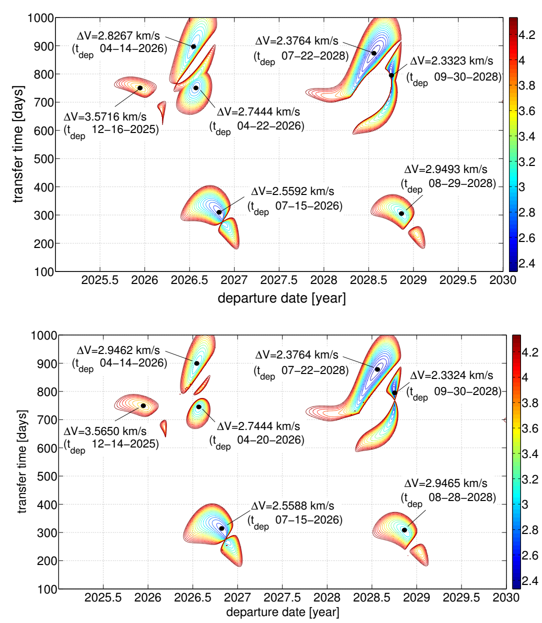

Fig. 2. Numerical (top) and analytical (bottom) pork-chop plot for the arrival excess velocity (ΔV) of an Earth-Mars transfer.

although the accuracy of the delta-V and launch date near minimum condition is not as remarkable as in the Mars case, especially for the C_3 plot. This is due to the low phasing efficiency (as low as $\sim$67%) of some of the best solutions found. In any case, the error remains within the limit of a few per-

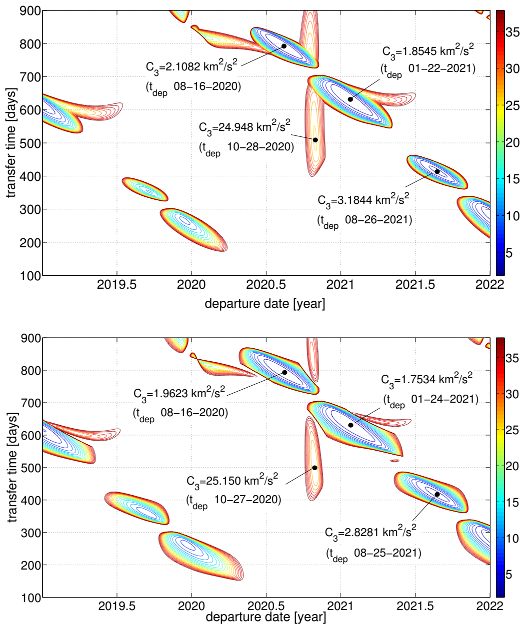

Fig. 3. Numerical (top) and analytical (bottom) pork-chop plot for the injection energy (C_3) of an Earth-Didymos transfer.

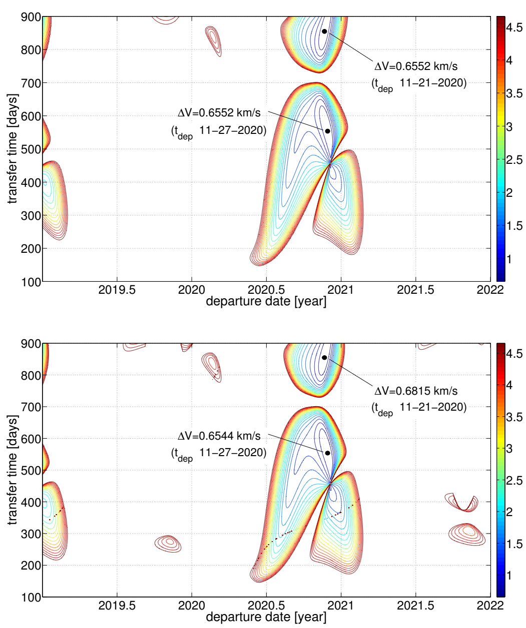

Fig. 4. Numerical (top) and analytical (bottom) pork-chop plot for the arrival excess velocity (ΔV) of an Earth-Didymos transfer.

cent, which may still be acceptable at a preliminary mission design stage.

The computational speed reduction factor in this case is 4.0, 5.3, 11.1 with respect to the method of Arora-Russell, Izzo and Gooding, respectively. The difference when compared to the Mars case is due to the problem dependency of the numerical methods (the computational time of the analytical method is virtually constant). For problems involving multi-rev solutions with a high number of revolutions the difference in performance may be substantial.

3. CONCLUSIONS

A novel approximate analytical solution of the Lambert's Targeting Problem (LTP) was presented and investigated. Although not yet extensively tested, the solution appears to be able to reproduce the most interesting parts of a generic pork-chop plot for interplanetary transfer trajectories. In particular, it offers remarkably good accuracy (considering its analytical character) in both estimated delta-V and launch date at local minimum points. As for computational speed, the proposed solution is at least 4 times faster than the most efficient Lambert solvers currently available and more than one order of magnitude faster than Gooding's method. A limited effort has been done in order to increase the efficiency of its implementation, which opens interesting possibilities for the future. The limited accuracy of the method, which is obviously its main drawback, should be traded off with the need of computationally effective solutions when dealing with massive and highly complex optimization problems. Many of these aspects will need to be studied extensively in the future.

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