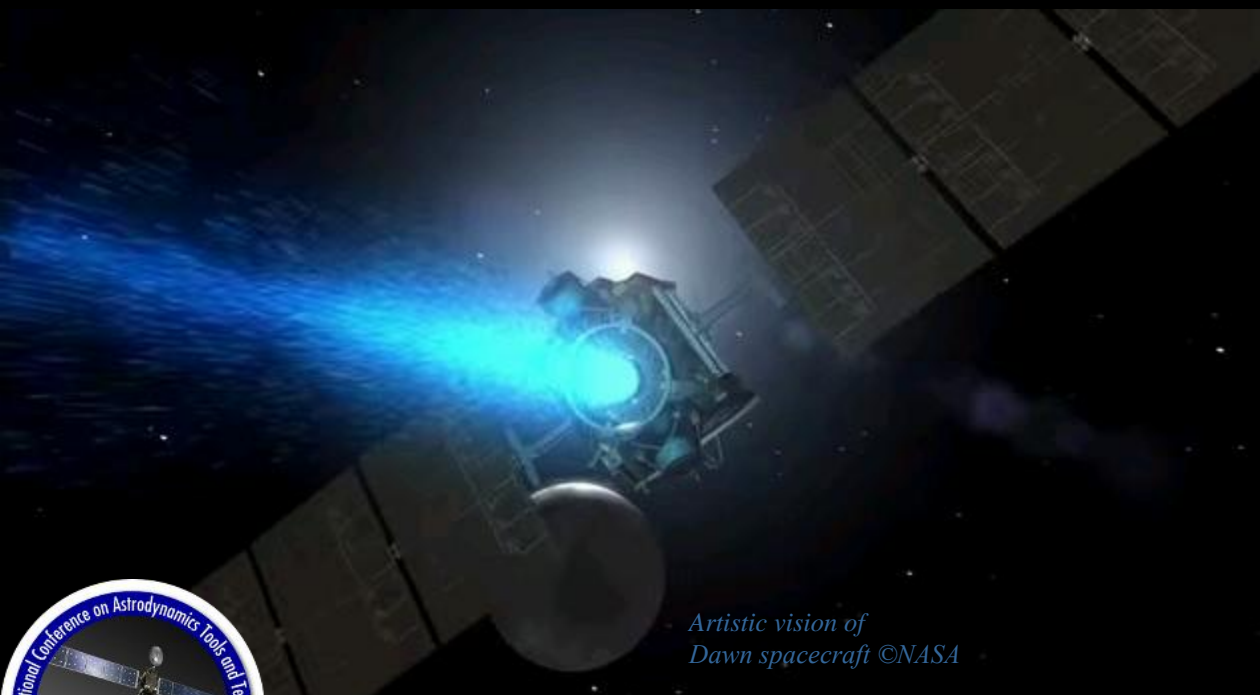


OPTIMIZATION OF LOW THRUST MULTI- REVOLUTION ORBITAL TRANSFERS USING THE METHOD OF DUAL NUMBERS



*Artistic vision of
Dawn spacecraft ©NASA*

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THE MAIN ASPECTS OF THE INTERORBITAL FLIGHTS OPTIMIZATION PROBLEM OF THE SPACECRAFT WITH EPS

The main aspects of the motion of the SC with EPS in the vicinity of the Earth:

- Thrust acceleration is small compared with gravity acceleration in the vicinity of the central body
- The action of different types of perturbation
- Large duration of the interorbital flight
- Large angular distance of the interorbital flight

The model which describes the motion of the spacecraft with EPS is rather complicated

Optimization problem:

- Using the *indirect* optimization method – Pontryagin's maximum principle

Necessity of solving boundary value problem

CALCULATING THE DERIVATIVES IN SOLVING THE OPTIMIZATION PROBLEMS FOR ORBITAL TRANSFER

It is required to define precise derivatives:

- derivatives of the residual function of the boundary conditions for solving boundary value problem of the maximum principle
- calculating the right hand side of the system of differential equations of optimal motion of the spacecraft with EP taking into account the different perturbations

It is usually necessary to use a fairly complex computational procedures to calculate the required derivatives, as the accuracy of the calculations depends on the convergence of boundary value problem of the maximum principle, as well as the accuracy of modeling of the spacecraft motion within the solution of the optimization problem.

MAIN METHODS FOR NUMERICAL THE DERIVATIVES OF THE FUNCTIONS

Finite differences methods

Forward differences:

$$\frac{df(x)}{dx} = \frac{f(x+h) - f(x)}{h} + O(h)$$

truncation error $O(h)$

Central differences:

$$\frac{df(x)}{dx} = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

truncation error $O(h^2)$

Reduction the truncation error is depend on the step reduction of h

$$h \rightarrow 0$$

Computational Aspects:

- There is a *subtraction cancellation error* associated with the calculation of the difference between the two close numbers. The nature of the error lies in the fact that the calculation is carrying out with the same relative precision.
- With decreasing the step size h , error caused by the difference between two close numbers increases and after that becomes dominant over the truncation error.
- For efficient use of finite difference methods it is required to find a compromise between these two errors in selecting the step size h .

MAIN METHODS FOR NUMERICAL THE DERIVATIVES OF THE FUNCTIONS

Complex step method

The expansion of $f(x)$ function in a Taylor series in the close neighborhood of x for the increment of its argument in the complex domain:

$$f(x + ih) = f(x) + ih \frac{f'(x)}{1!} - h^2 \frac{f''(x)}{2!} - ih^3 \frac{f'''(x)}{3!} + \dots + o(h^3)$$

The imaginary part of the expansion :

$$\text{Im}(f(x + ih)) = h f'(x) - h^3 \frac{f'''(x)}{3!} + \dots + o(h^3)$$

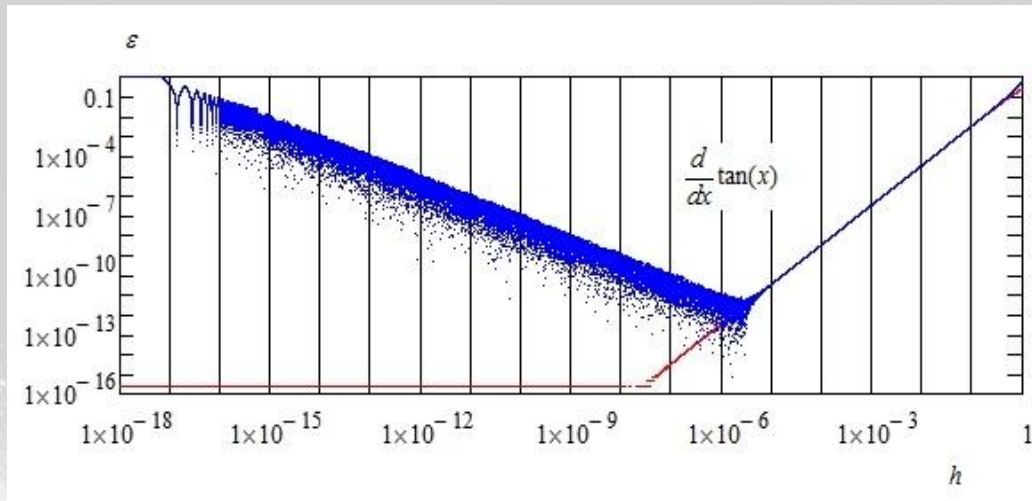
The expression for the derivative:

$$\frac{df(x)}{dx} = \frac{\text{Im}(f(x + ih))}{h} + O(h^2) \quad \boxed{h \rightarrow 0}$$

Computational Aspects:

- Truncation error when using the method of the complex step is the same order as when using the central differences– $O(h^2)$.
- Unlike difference methods, we can achieve a significant reduction in its value by setting an arbitrarily small step, thus rounding error when the difference between two numbers close does not occur due to the lack of subtraction in the expression for the derivative
- For some functions, *subtraction cancellation error* can occur because of the smallness of the imaginary parts - in the calculation functions in the complex domain

MAIN METHODS FOR NUMERICAL THE DERIVATIVES OF THE FUNCTIONS



$$\frac{d}{dx} \tan(x) = \tan(x)^2 + 1$$

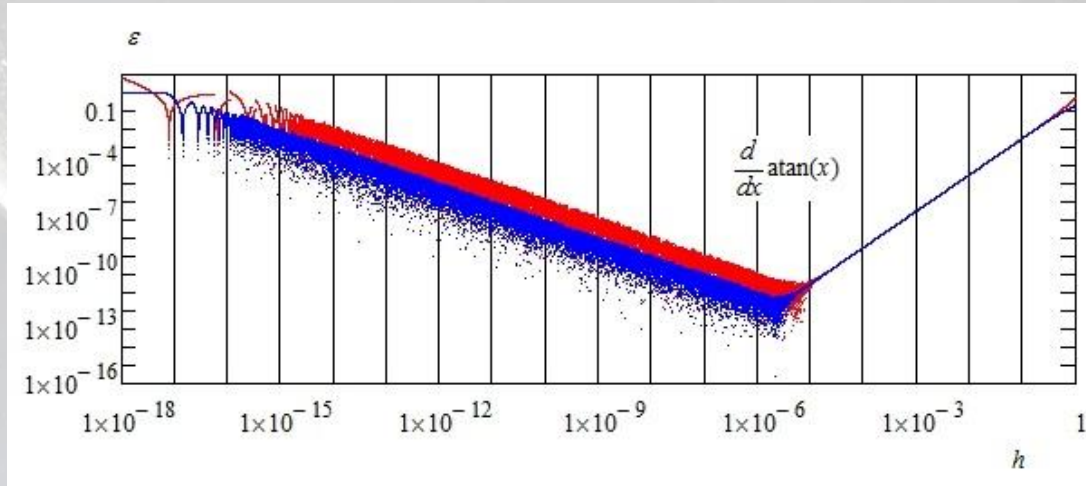
Comparison of the relative error of calculating the derivative of $\tan(x)$ using the central differences and method of comprehensive step by step size h .

- With decreasing the step size relative error of calculating the derivative using the central difference (blue line) first decreases monotonically, as long as the prevalent component of the truncation error while achieving a minimum. Then, begin to dominate component of the subtraction cancellation error, which leads to a monotonic increase in the relative error in the determination of the derivative with a further reduction of step size h .
- The relative error when using the method of complex steps (red line) decreases monotonically with the step size, reaching a value corresponding to the relative accuracy of the calculations.

MAIN METHODS FOR NUMERICAL THE DERIVATIVES OF THE FUNCTIONS

- Imaginary part of the expression for the derivative is small with a small h .
- The process of calculating certain functions in the complex domain itself can be the difference representation

$$\frac{df(x)}{dx} = \frac{\text{Im}(f(x+ih))}{h} + O(h^2)$$



$$\frac{d}{dx} \arctg(x) = \frac{1}{1+x^2}$$

$$\arctg(x+ih) = -\left(\frac{i}{2}\right) \ln\left(\frac{1+ix-h}{1-ix+h}\right)$$

The difference between the two close values

Comparison of the relative error of calculating derivative $\arctan(x)$ function using the central differences and method of comprehensive step by step size h .

Behavior of relative error as a function of the length of the pitch in the case provided by the same complex step method (red line) and the central differences (blue line), due to the appearance of subtraction cancellation errors.

MAIN METHODS FOR NUMERICAL THE DERIVATIVES OF THE FUNCTIONS

Disadvantages of the complex step method:

- Calculation of certain functions in the complex domain, as well as a number of computational operations is very costly in terms of computing resources.
- Small order of the imaginary parts of the complex step method reduces the efficiency of its practical application, for example, to calculate the sensitivity matrix in the solution of nonlinear boundary value problem by *continuation method*.

MAIN METHODS FOR NUMERICAL THE DERIVATIVES OF THE FUNCTIONS

The method of dual numbers

Dual number:

$$\langle x, x' \rangle = x + dx', \quad x, x' \in \mathbf{R},$$

x – real part,
x' – dual part, $d^2 = 0$

Isomorphism with space \mathbf{R}^2 , which is similar to complex numbers

The dual number as a vector with the dual part s– the form of the number:

$$\langle x, x', x'', \dots \rangle = x + x'd_1 + x''d_2 + \dots;$$

$x, x', x'', \dots \in \mathbf{R},$
x – real part,
x', x'', \dots – elements of vector of the dual parts,
 $d_1^2 = d_2^2 = \dots d_n^2 = 0$ u $d_i d_j = 0$ при $i \neq j$.

MAIN METHODS FOR NUMERICAL THE DERIVATIVES OF THE FUNCTIONS

Calculation of derivatives

Taylor series expansion:

$$f(x + dh) = f(x) + dh \frac{f'(x)}{1!} + (dh)^2 \frac{f''(x)}{2!} + (dh)^3 \frac{f'''(x)}{3!} + \dots$$

$$f(x + dh) = f(x) + dh f'(x)$$

Set $h=1$:

$$\frac{df(x)}{dx} = Dp(f(x+1d)), \quad f(x) = Rp(f(x+1d))$$

Rp – Real Part
 Dp – Dual Part

- Truncation error in calculating derivatives by dual numbers is zero.
- In the expression of the derivative there is no difference, which is similar to the method of complex steps.
- The orders of the real and the dual part of the right hand side of the expression are the same, or may always be selected as such an appropriate assignment value h . Therefore, unlike the method of complex step, we do not face in this case with subtraction cancellation error.
- It is possible in the process of computation automatically obtain the required values of derivatives with relative precision equal to the precision of computing functions.

MAIN METHODS FOR NUMERICAL THE DERIVATIVES OF THE FUNCTIONS

Calculation of derivatives of the function with several variables.

Example for the function of two variables:

$$\left. \begin{aligned} y &= f(x_1, x_2), \\ x_1 &+ x_1' d_1 + 0 d_2, \\ x_2 &+ 0 d_1 + x_2'' d_2, \\ y &= \langle y, y', y'' \rangle = y + y' d_1 + y'' d_2, \\ x_1' &= x_2'' = 1, \\ f(x_1, x_2) &= Rp(y), \\ \frac{\partial f}{\partial x_i} &= Dp_i(y), i = 1, 2. \end{aligned} \right\}$$

The dimension of the *vector dual part* is determined by the number of function arguments.

USING THE METHOD OF DUAL NUMBERS IN SOLVING OPTIMIZATION PROBLEM OF ORBITAL TRANSFER

- The method of dual numbers is used to automatic differentiation
- Dual numbers with a single dual part are used to compute the sensitivity matrix for solving the nonlinear boundary value problem of the maximum principle by continuation method.
- Dual numbers with vector dual part are proposed to use in solving the optimization problem of the multi-revolution orbital transfer, taking into account the most complete perturbation model. In this case it is assumed that the automatic calculation of the right hand sides of the equations of optimal motion of spacecraft by calculating the optimal Hamiltonian - by virtue of the canonical formalism of the maximum principle.

MODEL OF THE SPACECRAFT MOTION

SC movement with EP is described by the following system of differential equations in the equinoctial elements:

$$\frac{dp}{dt} = 2 \frac{p^{3/2}}{b_1} T$$

$$\frac{de_x}{dt} = \frac{p^{1/2}}{b_1} \left(\left((1+b_1) \cos(l) + e_x \right) T + b_1 \sin(l) S - e_y b_2 W \right)$$

$$\frac{de_y}{dt} = \frac{p^{1/2}}{b_1} \left(\left((1+b_1) \sin(l) + e_y \right) T - b_1 \cos(l) S + e_x b_2 W \right)$$

$$\frac{di_x}{dt} = \frac{1}{2} \frac{p^{1/2}}{b_1} b_3 \cos(l) W$$

$$\frac{di_y}{dt} = \frac{1}{2} \frac{p^{1/2}}{b_1} b_3 \sin(l) W$$

$$\frac{dl}{dt} = \frac{b_1^2}{p^{3/2}} + p^{1/2} \frac{b_2}{b_1} W$$

$$b_1 = 1 + e_x \cos(l) + e_y \sin(l), \quad b_2 = i_x \sin(l) - i_y \cos(l), \quad b_3 = 1 + i_x^2 + i_y^2,$$

$$e_x = e \cos(\Omega + \omega), \quad e_y = e \sin(\Omega + \omega), \quad i_x = \tan(i/2) \cos(\Omega), \quad i_y = \tan(i/2) \sin(\Omega),$$

$$l = \Omega + \omega + \nu.$$

$$S = S_a + \sum_j S_{\phi_j},$$

$$T = T_a + \sum_j T_{\phi_j},$$

$$W = W_a + \sum_j W_{\phi_j}$$

S_a, T_a, W_a – thrust acceleration components

$S_{\phi_j}, T_{\phi_j}, W_{\phi_j}$ – perturbations components

Limited thrust:

$$S_a = \delta (P/m) \sin(\theta) \cos(\psi),$$

$$T_a = \delta (P/m) \cos(\theta) \cos(\psi),$$

$$W_a = \delta (P/m) \sin(\psi)$$

P - thrust, m – mass of SC; θ, ψ – pitch and yaw angle, $\delta \in \{0, 1\}$;
 $\theta(t) \in [0, 2\pi), \quad \psi(t) \in [-\pi/2, \pi/2]$

$$dm/dt = -\delta(P/w) \quad \text{OT}$$

$$dm/dt = -(m^2 a^2) / (2N_r) \quad \text{OM}$$

Power limited

$$S_a = a_S,$$

$$T_a = a_T,$$

$$W_a = a_W$$

$$a = \|a_S, a_T, a_W\|$$

w – effective exhaust velocity

N_r - Reactive Power

FORMULATION OF THE OPTIMIZATION PROBLEM

Objective function of the problem is set at a fixed interval of time:

$$J = -m(t_f) \rightarrow \min, \quad t \in [t_0, t_f] \rightarrow \text{fixed} \quad - \text{maximization of final mass of SC (EPS for both engines models)}$$

Hamilton-Pontryagin function:

$$H = H_1 + H_2 + H_3$$

$$H_1^{OT} = \delta(P/m)(a_1 \sin(\vartheta) \cos(\psi) + a_2 \cos(\vartheta) \cos(\psi) + a_3 \sin(\psi)) - \psi_m \delta(P/w)$$

$$H_1^{OM} = a_s a_1 + a_T a_2 + a_w a_3 - \psi_m \frac{m^2 (a_s^2 + a_T^2 + a_w^2)}{2N_r}$$

$$H_2 = \psi_l \frac{b_1^2}{p^{3/2}}, \quad H_3 = a_1 \sum_j S_{\Phi_j} + a_2 \sum_j T_{\Phi_j} + a_3 \sum_j W_{\Phi_j},$$

$$a_1 = p^{1/2} (\psi_{e_x} \sin(l) - \psi_{e_y} \cos(l))$$

$$a_2 = \frac{p^{1/2}}{b_1} (2 p \psi_p + \psi_{e_x} ((1+b_1) \cos(l) + e_x) + \psi_{e_y} ((1+b_1) \sin(l) + e_y))$$

$$a_3 = \frac{p^{1/2}}{b_1} \left(b_2 (\psi_{e_y} e_x - \psi_{e_x} e_y + \psi_l) + \frac{b_3}{2} (\psi_{i_x} \cos(l) + \psi_{i_y} \sin(l)) \right)$$

$$\Psi = (\psi_p, \psi_{e_x}, \psi_{e_y}, \psi_{i_x}, \psi_{i_y}, \psi_l), \psi_m \in BV([t_0, t_f], \mathbf{R}^7)$$

$$\mathbf{x} = (p, e_x, e_y, i_x, i_y, l), m \in AC([t_0, t_f], \mathbf{R}^7)$$

Control:

$$\mathbf{u}_{OT} \in U_{OT} \subset \mathbf{R}^3$$

$$U_{OT} \equiv \left\{ (\vartheta, \psi, \delta) \mid \vartheta \in [0, 2\pi), \right. \\ \left. \psi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right], \delta \in [0, 1] \right\}$$

$$\mathbf{u}_{OM} \in U_{OM} \subseteq \mathbf{R}^3$$

$$U_{OM} \equiv \{(a_s, a_T, a_w) \mid a_j \in \mathbf{R}\}$$

FORMULATION OF OPTIMIZATION PROBLEM

Terminant:

$$\ell = \alpha_0(-m(t_f)) + \beta_0 \mathbf{f}_0(\mathbf{x}(t_0)) + \beta_k \mathbf{f}_k(\mathbf{x}(t_f)) + \beta_{m0}(m(t_0) - m_0)$$

numbers α_0, β_{m0} and vectors β_0, β_k – Lagrange multipliers

$\mathbf{f}_0(\mathbf{x}(t_0)), \mathbf{f}_k(\mathbf{x}(t_f))$ - vector function, which defines parameters of the initial and final orbit

m_0 - initial mass of the spacecraft

Optimal control:

OT problem

$$\begin{pmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \\ u_4^0 \end{pmatrix} = \begin{cases} a_2 / \sqrt{a_1^2 + a_2^2}; \\ a_1 / \sqrt{a_1^2 + a_2^2}; \\ \sqrt{a_1^2 + a_2^2} / \sqrt{a_1^2 + a_2^2 + a_3^2}; \\ a_3 / \sqrt{a_1^2 + a_2^2 + a_3^2}. \end{cases} \quad \delta^0 = \begin{cases} 1, & \Pi > 0; \\ 0, & \Pi < 0; \\ \forall \delta \in (0, 1), & \Pi = 0. \end{cases} \quad \Pi = \frac{P}{m} \sqrt{a_1^2 + a_2^2 + a_3^2} - \psi_m \frac{P}{w}$$

$$u_1^0 = \cos(\theta), u_2^0 = \sin(\theta), u_3^0 = \cos(\psi), u_4^0 = \sin(\psi)$$

OM problem

$$a_s^0 = \frac{N_r}{\psi_m m^2} a_1, a_r^0 = \frac{N_r}{\psi_m m^2} a_2, a_w^0 = \frac{N_r}{\psi_m m^2} a_3$$

FORMULATION OF OPTIMIZATION PROBLEM

Adjoint equations for OT and OM problem:

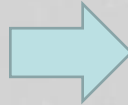
$$\left. \begin{aligned} \frac{d\psi}{dt} &= -\frac{\partial H}{\partial \mathbf{x}} \\ \frac{d\psi_m}{dt} &= \delta \frac{P}{m^2} \sqrt{a_1^2 + a_2^2 + a_3^2} \end{aligned} \right\}$$

$$\left. \begin{aligned} \frac{d\psi}{dt} &= -\frac{\partial H}{\partial \mathbf{x}} \\ \frac{d\psi_m}{dt} &= \frac{N_r}{\psi_m m^3} (a_1^2 + a_2^2 + a_3^2) \end{aligned} \right\}$$

Transversality conditions:

$$\psi(t_0) = \frac{\partial \ell}{\partial \mathbf{x}} \Big|_{t_0}, \quad \psi(t_k) = -\frac{\partial \ell}{\partial \mathbf{x}} \Big|_{t_k},$$

$$\psi_m(t_0) = \frac{\partial \ell}{\partial m} \Big|_{t_0}, \quad \psi_m(t_k) = -\frac{\partial \ell}{\partial m} \Big|_{t_k}$$



The consequence of the maximum principle conditions:

$$\psi_m(t_f) = 1$$

Optimal Hamiltonian in the case of OT task:

$$H^{OT} = \delta(P/m) \sqrt{a_1^2 + a_2^2 + a_3^2} - \psi_m \delta(P/w) + H_2 + H_3$$

Optimal Hamiltonian in the case of OM task:

$$H^{OM} = \frac{N_r}{2\psi_m m^2} (a_1^2 + a_2^2 + a_3^2) + H_2 + H_3$$

Homotopy between OM and OT tasks:

$$\left. \begin{aligned} \mathbf{x} &= (p, e_x, e_y, i_x, i_y, l)^T, \\ H^{OM} &= H_1^{OM} + H_2, \\ H^{OT} &= H_1^{OT} + H_2, \\ \dot{\mathbf{x}} &= \frac{\partial H^{OM}}{\partial \psi} + \tau \left(\frac{\partial H^{OT}}{\partial \psi} - \frac{\partial H^{OM}}{\partial \psi} \right), \\ \dot{\psi} &= \tau \left(\frac{\partial H^{OM}}{\partial \mathbf{x}} - \frac{\partial H^{OT}}{\partial \mathbf{x}} \right) - \frac{\partial H^{OM}}{\partial \mathbf{x}}, \\ \dot{\psi}_m &= \tau \left(\frac{\partial H^{OM}}{\partial m} - \frac{\partial H^{OT}}{\partial m} \right) - \frac{\partial H^{OM}}{\partial m}. \end{aligned} \right\}$$

APPLICATION OF THE DUAL NUMBERS METHOD IN CONTINUATION METHOD

The system of nonlinear equations corresponding to the boundary value problem of the maximum principle:

$$\mathbf{f}(\mathbf{p}) = 0, \mathbf{f} : \mathbf{R}^n \rightarrow \mathbf{R}^n, \mathbf{p} \in \mathbf{R}^n. \quad (*)$$

The inner Cauchy problem :

$$\frac{d\mathbf{x}}{dt} = \varphi(\mathbf{x}, t, \tau), \mathbf{x}(t_0) = \mathbf{x}_0, t \in [t_0, t_1]. \quad \mathbf{x} : t \rightarrow \mathbf{R}^N, \varphi(\mathbf{x}, t, \tau) \in C^1(\mathbf{R}^N \times \mathbf{R} \times \mathbf{R})$$

Boundary conditions, according to (*):

$$\mathbf{K}(\mathbf{x}(t_0), \mathbf{x}(t_1), t_0, t_1, \tau) = 0, \mathbf{K} \in C^1(\mathbf{R}^{2N+3})$$



$$\mathbf{f}(\mathbf{p}, \tau) = 0$$



External Cauchy's problem

$$\left. \begin{aligned} \frac{d\mathbf{p}}{d\tau} &= - \left(\frac{\partial \mathbf{f}}{\partial \mathbf{p}} \Big|_{\mathbf{p}=\mathbf{p}(\tau)} \right)^{-1} \left(\mathbf{f}(\mathbf{p}_0) + \frac{\partial \mathbf{f}}{\partial \tau} \right), \\ \mathbf{p}(0) &= \mathbf{p}_0, \quad \tau \in [0, 1]. \end{aligned} \right\}$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{p}}, \quad \frac{\partial \mathbf{f}}{\partial \tau}$$

-necessary to define

APPLICATION OF THE DUAL NUMBERS METHOD IN CONTINUATION METHOD

For solving the external Cauchy's problem:

- It is supposed to build the computational scheme using dual numbers with a single dual part.
- At one stage of the external integration the relative error in the calculation of the partial derivatives will be of the order is not greater than given relative accuracy of internal integration.
- By increasing the accuracy of external integration, it is possible to reduce the "initial data" error (at each step of the external integration) for internal integration, and as a result - total error in the calculation of the sensitivity matrix.
- By increasing the accuracy of both internal and external integration it is possible to achieve a more sustainable process with a significantly smaller number of discarded steps than in the case of using the method of complex steps to calculate partial derivatives

APPLICATION OF THE DUAL NUMBERS METHOD IN CONTINUATION METHOD

To calculate the sensitivity matrix elements it is required to carry out internal integration over the field of dual numbers. Due to the complexity of numerical realization of such integration, it is advisable to use the following (equivalent to it) calculation scheme:

$$\left. \begin{aligned} \mathbf{y} \in \mathbf{R}^{2N}, \mathbf{x}^d \in \text{DualNumber}^N \\ y_i = \begin{cases} Rp(\varphi(\mathbf{x}^d)_i), i = 1, 2..N \\ Dp(\varphi(\mathbf{x}^d)_{i-N}), i = (N+1)..2N \end{cases} \\ \dot{\mathbf{y}} = \mathbf{y}, \rightarrow \mathbf{y1} \end{aligned} \right\}$$

- Right hand side of the system of differential equations are calculated in the dual representation;
- \mathbf{y} – the new expanded system of differential equations of order $2N$ – is considered in the real domain.

The diagram above is repeated at each step of the internal integration. Integrating internal Cauchy problem and calculating the dual representation of the residuals, together with their values at the same time we get the sensitivity matrix elements.

NUMERICAL RESULTS

OT-problem without perturbations

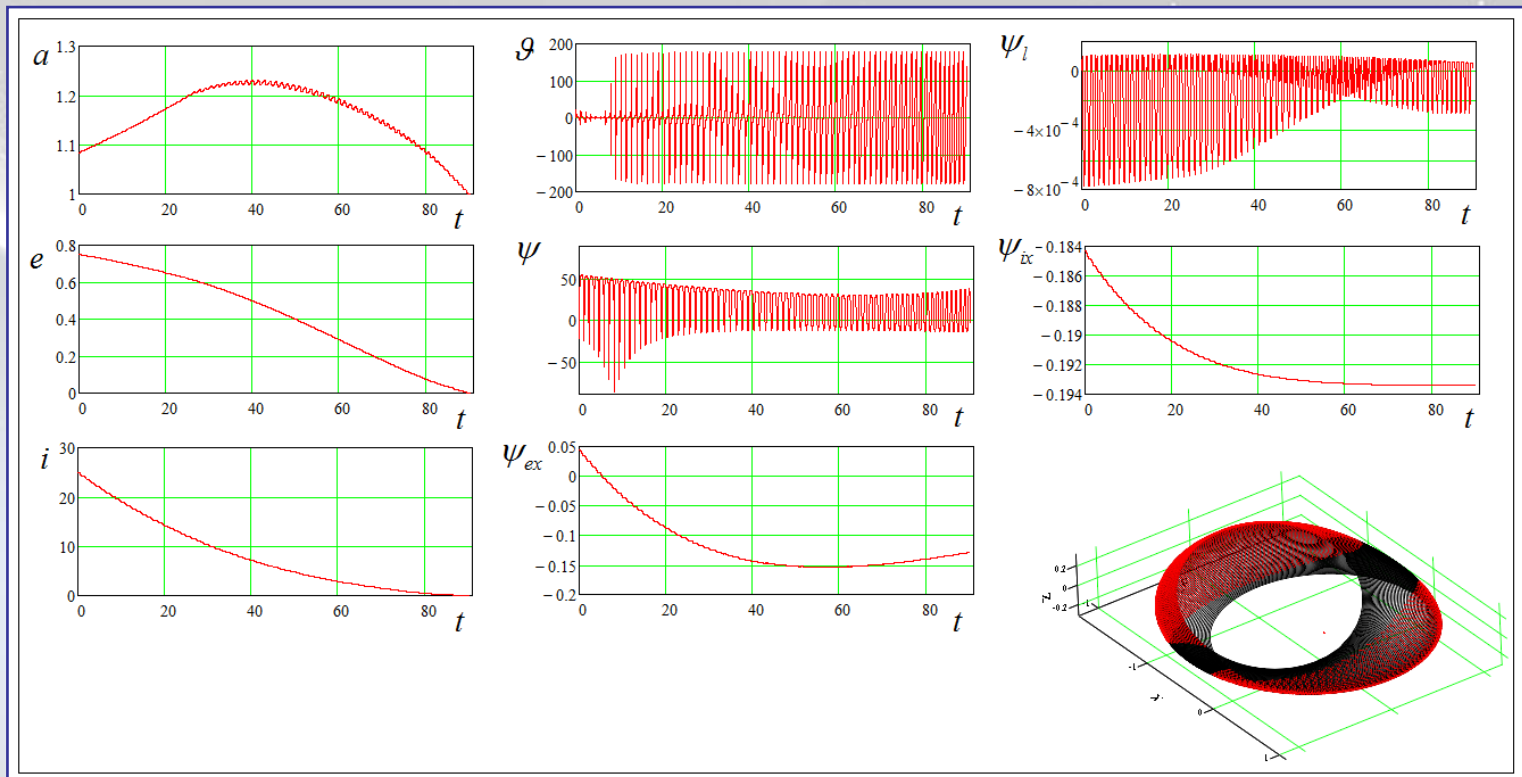
The initial orbit has the following characteristics:

$p=20000 \text{ km}$, $e=0.75$, $i=25 \text{ deg}$, $\Omega=\omega=0 \text{ deg}$; $v=200 \text{ deg}$.

The angular distance of the flight is not fixed, the duration of the flight is 90 days.

SC characteristics are follows: initial mass of 1320 kg, the thrust EPS - 0.4 N, the specific impulse – 1500 s.

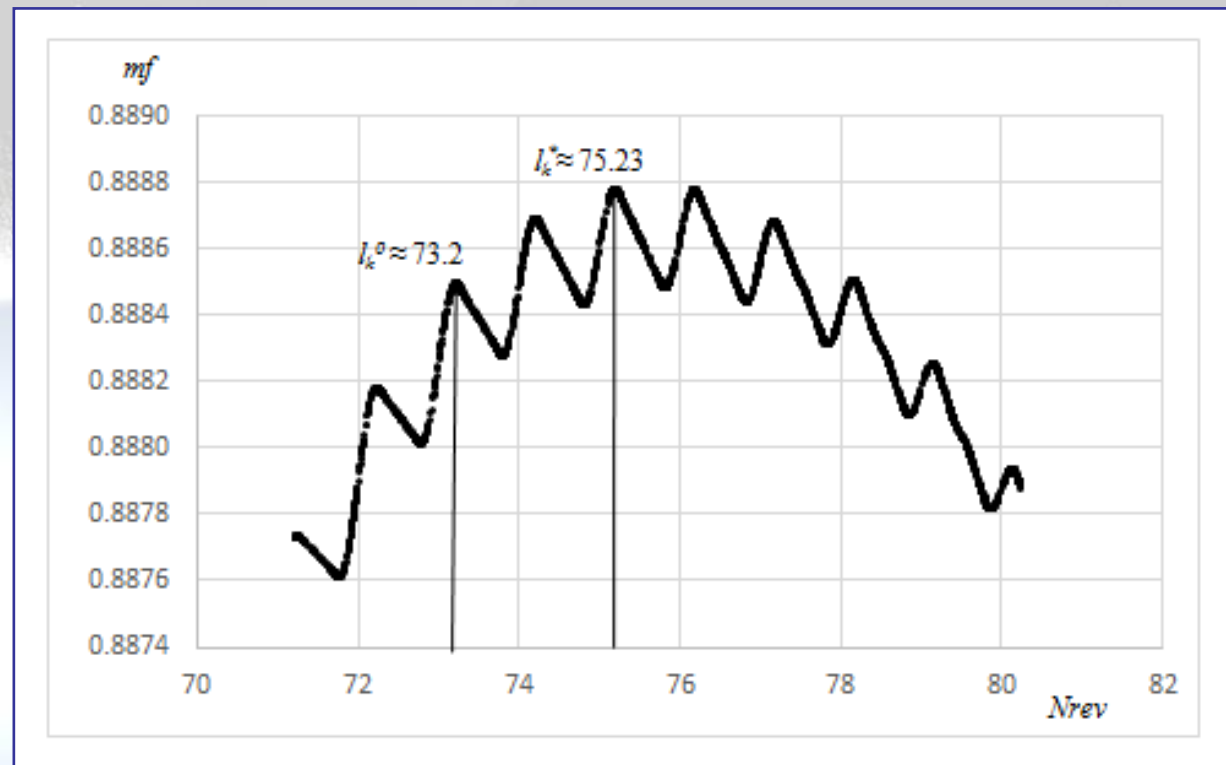
The final mass of the SC: 0.88865 [rel.], or 1173.18 kg.



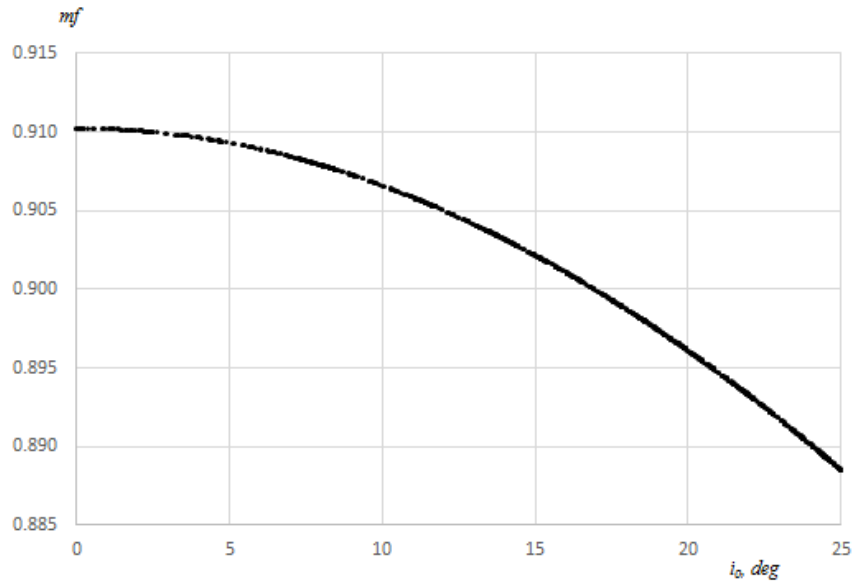
NUMERICAL RESULTS

- Formalism of the continuation method was used in the analysis of the trajectory in terms of determining the optimum angular distance on the trajectory between these orbits.
- The resulting solution of the problem has been continued by the angular distance from the point corresponding to the final value of the true longitude l_k^0 of the given solution with two revolutions in the backward direction and seven revolutions in the forward direction.
- This has been implemented as follows: the condition $\psi_f(t_f)=0$ was used in the boundary conditions of the problem instead of the condition $l(t_f)-(l_k^0 \pm \tau(2\pi N))$, where N - number of whole revolutions, τ - parameter of the continuation.
- The modified Newton homotopy is used for continuation on the boundary conditions. Each step of the method of continuation of solution of the problem corresponds to the current value of the angular distance of $l_k(\tau)=l_k^0 \pm \tau(2\pi N)$.

Each step of the method of continuation of solution of the problem corresponds to the current value of the angular distance of $l_k(\tau)=l_k^0 \pm \tau(2\pi N)$.



NUMERICAL RESULTS



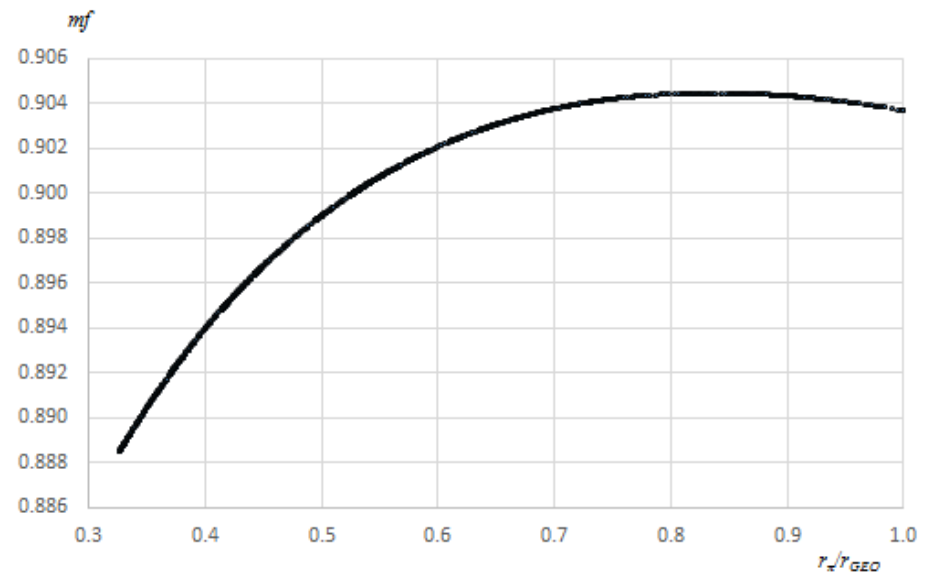
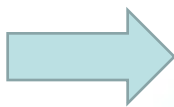
The continued solution by boundary conditions:

- *by inclination of the initial orbit*



The continued solution by boundary conditions:

- By radius of perigee of the initial orbit



THE PROCEDURE FOR CALCULATING OF THE RIGHT HAND SIDE OF OPTIMAL MOTION EQUATIONS BY DUAL NUMBERS METHOD

This procedure is based on the use of the canonical formalism of the maximum principle: optimum Hamiltonian completely defines the system, describing the optimal process. It takes only one calculation of optimal Hamiltonian in the dual representation for each access to the right parts of the numerical integration of the system of equations managed spacecraft motion. The procedure can be described by the following scheme:

$$i = 1, 2 \dots n, j = 1, 2 \dots 2 \cdot n,$$

$$\mathbf{x} = (x_1, \dots, x_i, \dots, x_n)^T,$$

$$\boldsymbol{\psi} = (\psi_1, \dots, \psi_i, \dots, \psi_n)^T,$$

$$x_i = x_{i,0} + \sum_{j=1}^{2n} x_{i,j} \cdot d_j,$$

$$\psi_i = \psi_{i,0} + \sum_{j=1}^{2n} \psi_{i,j} \cdot d_j,$$

$$\left. \begin{array}{l} x_{i,j} = 1, j = i \\ x_{i,j} = 0, j \neq i \end{array} \right\}, \quad \left. \begin{array}{l} \psi_{i,j} = 1, j = i + n \\ x_{i,j} = 0, j \neq i + n \end{array} \right\},$$

$$\dot{x}_i = Dp_{n+i}(H^0(\boldsymbol{\psi}, \mathbf{x})),$$

$$\dot{\psi}_i = -Dp_i(H^0(\boldsymbol{\psi}, \mathbf{x})),$$

$$H^0 = Rp(H^0(\boldsymbol{\psi}, \mathbf{x})).$$

$\mathbf{x}, \boldsymbol{\psi}$ - phase vector and adjoints vector of the system (dual representation);
 $H^0(\boldsymbol{\psi}, \mathbf{x})$ - dual representation of the of the optimal Hamiltonian

NUMERICAL EXAMPLE

OT-problem with perturbations

The following perturbations acting on the spacecraft are taking into account :

from the moon and Sun, the non-centrality of the gravitational field of the Earth. When calculating the lunisolar perturbations, coordinates of the Moon and the Sun were determined by using DE405 ephemeris. Centrifugal accelerations of the gravitational field of the non-centrality are calculated as derivatives of the geopotential represented as an expansion in terms of spherical functions, associated with the Earth ITRS coordinate system. We take into account the *precession* and *nutation* of the Earth's axis and the movement of the pole according to the model IAU 2006 / 2000A. The necessary calculations were performed using the SOFA IAU program. To calculate the geopotential model was used Earth's gravitational field EGM-96.

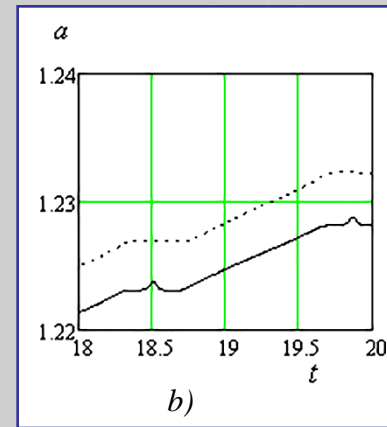
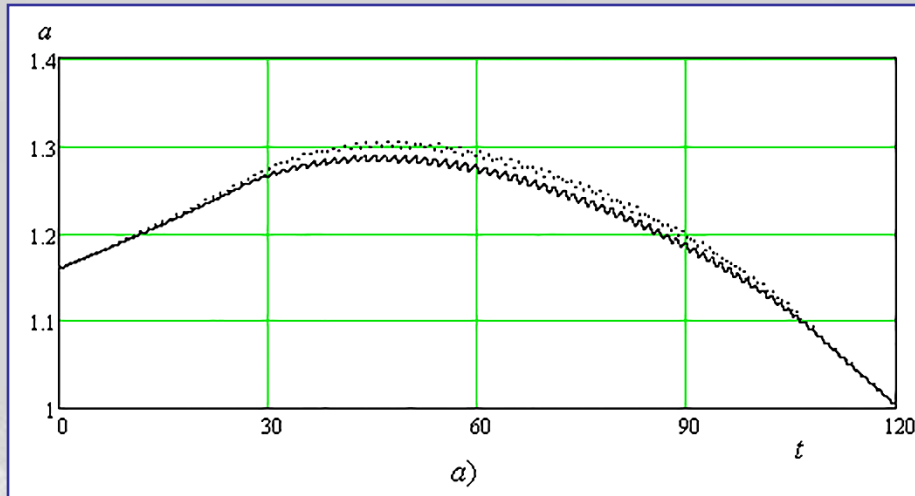
Parameters of the initial orbit of the spacecraft are following: perigee altitude - 5000 km, apogee altitude - 80000 km, the $i=25$ deg, $\Omega=\omega=0$ deg, $v=200$ deg. The initial mass of the SC - 2500 kg thrust - 0.56N, the specific impulse - 1640s. The duration of the flight to GEO - 120 days.

Geopotential - 4×4 .

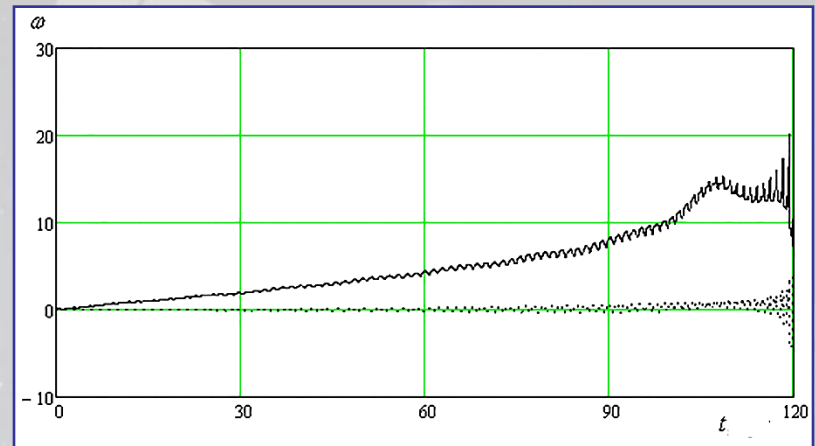
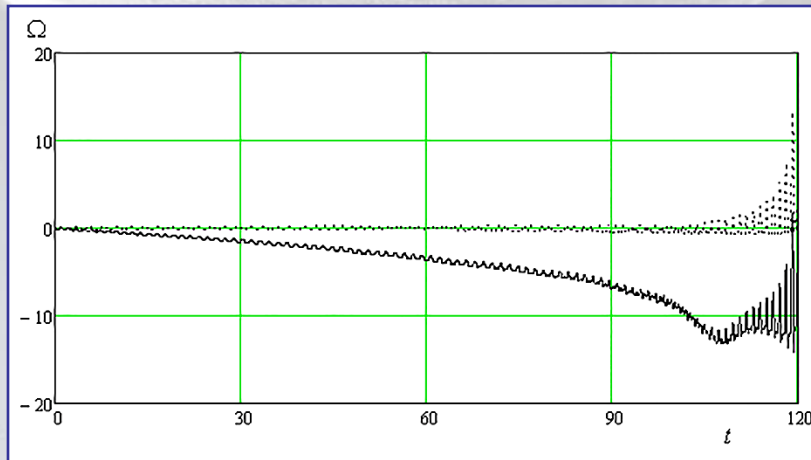
Date of start corresponds to December 26, 2015 1:00 32 minutes UTC.

Dimensionless mass of SC was found as 0.89493 which is slightly less compared to the same problem without taking into account the perturbations - 0.89540.

NUMERICAL EXAMPLE



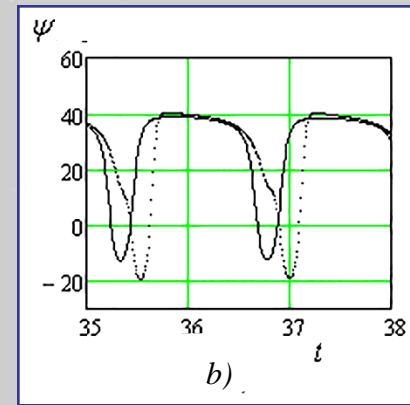
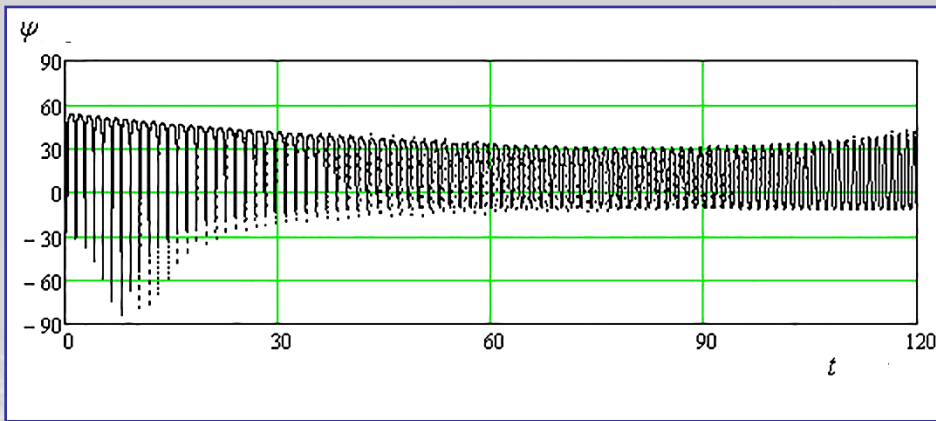
Evolution of the semimajor axis of the orbit of the spacecraft [dim] : a) for the entire duration of the trip; b) in the range from 18 to 20 days.



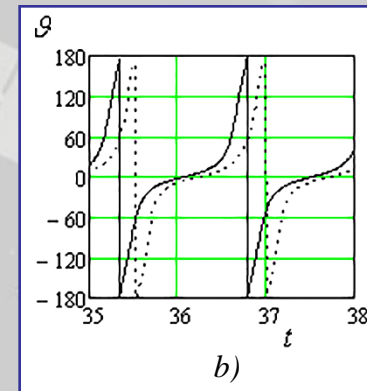
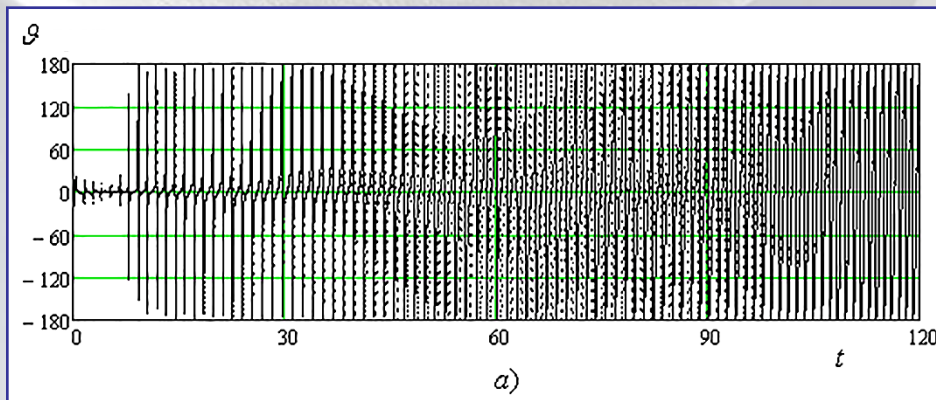
The evolution of the ascending node and argument of pericenter

The solid line corresponds to the perturbed solution and point corresponds to unperturbed.

NUMERICAL EXAMPLE



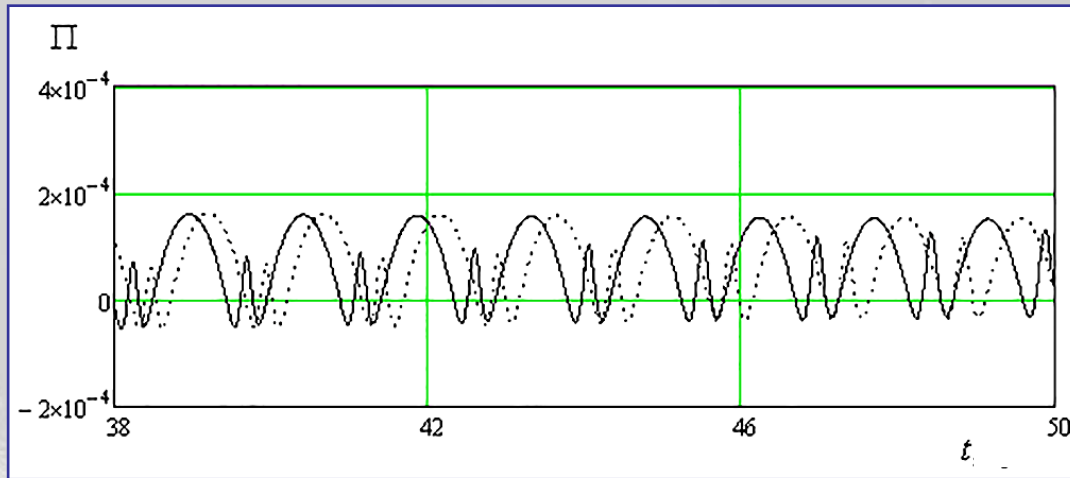
Dependence of change of yaw angle: *a)* the entire flight time; *b)* in the range from 35 to 38 days.



Dependence of change of the pitch angle *a)* all trip time; *b)* in the range from 35 to 38 days.

The solid line corresponds to the perturbed solution and point corresponds to unperturbed.

NUMERICAL EXAMPLE



Switching function of the EPS – Π [dim] on the spacecraft flight duration ranging from 38 to 50 days.

The solid line corresponds to the perturbed solution and point corresponds to unperturbed.

CONCLUSION

- We analyzed two possible aspects of using the dual numbers method for solving optimization problems for the spacecraft orbital transfer with EP.
- The first of these corresponds to the use of dual numbers with a single dual part to calculate the elements of sensitivity matrix in the solution of nonlinear boundary value problem of the maximum principle by the continuation method.
- The second aspect involves the use of dual numbers method with the vector of the dual parts for the purpose of calculating the right hand sides of the optimal motion of the system in the case of solving the problem of optimizing the spacecraft orbital transfer, taking into account the complete model of perturbations.
- The results using the dual numbers method for the fuel optimal transfer problem with EP from initial elliptical orbit into geostationary orbit are presented and analyzed.