OPTIMIZATION OF LOW THRUST MULTI-REVOLUTION ORBITAL TRANSFERS USING THE METHOD OF DUAL NUMBERS

M.S. Konstantinov, I.A. Nikolichev, Min Thein
Moscow Aviation Institute, Moscow, Russia.
e-mail: mkonst@bk.ru, ianikolichev@gmail.com minnntheino@gmial.com

ABSTRACT

The possibility of using the method of dual numbers in automatic differentiation for solving optimization problems of the low-thrust multi-revolution orbital transfers is considered. Traditionally the motion equations for the spacecraft with electric propulsion for multi-revolution orbital transfers are written in osculating elements or their modifications which exclude the special features in the right-hand sides of differential equations. These right-hand sides of the equations become especially complicated when different perturbations influencing the spacecraft movement are taken into account. Within the formalism of the Pontryagin maximum principle the right-hand sides of the optimal motion equations for the adjoints equations are quite complicated which results in some difficulties in solving optimization problems. Therefore the use of dual numbers method in numerical differentiation of optimal Hamiltonian for calculating the right-hand sides of the optimal motion equations of the spacecraft is effective. Another aspect of using the dual numbers method for numerical differentiation is to calculate the sensitivity matrix when solving boundary value problem corresponding to the optimal control problem. In this case, using dual numbers method allows obtaining the accurate sensitivity matrix. When using the continuation method for solving boundary value problem it helps to improve the convergence and to significantly reduce the number of steps for the external integration of Cauchy problem. The numerical results for optimal multi-revolution orbital transfer from the arbitrary initial orbit into the geostationary orbit are presented.

Index Terms **-** electric propulsion, dual numbers, orbital transfer, trajectory optimization, low thrust

1. INTRODUCTION

The use of the electric propulsion (EP) in the various maneuvers in the vicinity of the Earth usually has the complicated control structure and transfer trajectory is characterized by large numbers revolutions because of the low level of acceleration acting on the spacecraft along the trajectory of the flight. The successful implementation spacecraft transfer requires the solutions of a number of problems related to the optimization of trajectories of spacecraft. Describing the motion of the spacecraft can be quite complicated and it is more difficult to use the classical mechanics for space flight with the low thrust optimization methods such as the use of the Pontryagin maximum principle [1,2]. For example, in solving optimization problems using the formalism of maximum principle including the perturbations acting on the spacecraft with electric propulsion, leads to considerable difficulties in the analytical form of the right hand sides of the adjoints system or it does not possible. In this paper, to avoid this problem we propose the use of dual numbers not only for automatic differentiation for the purpose of calculating the right sides of optimal spacecraft motion but also for solving nonlinear boundary value problem by continuation method.

2. APPLICATION OF DUAL NUMBERS METHOD FOR AUTOMATIC DIFFERENTIATION

The possibility of using the dual numbers method is considered for automatic differentiation in order to calculate the necessary derivatives for the trajectory optimization problems with electric propulsion. We briefly describe the dual numbers method and the methodology of its application to perform the required calculations. Let see the standard approach of the numerical definition of derivatives.

It is known that the basis for the numerical differentiation of functions is the finite difference methods. It is well established in the first place in terms of the simplicity of their construction and applications for their full definition are sufficient to use the Taylor series expansion of the interested function. Thus giving the increment of the function h at the point x and expanding it to the second-order terms in the Taylor series is easy to derive the following expression that determines the value of the derivative at a given point (for example, forward difference):

$$\frac{df(x)}{dx} = \frac{f(x+h) - f(x)}{h} + O(h). \qquad (1)$$

It can be seen from the above expression (1), the value of $O(h)$ determines the order of the truncation error caused by the finite approximation of an infinite process. For forward differences the order of the error is not less than the step size h. It follows that we can reduce the magnitude of truncation

error selecting the step size as less as possible. However, really small values of the step size *h*, we just approach to the difference between the two close values (which is typical for all the difference methods in general). In here, there is another error for calculation of the derivative that is the rounding error due to the difference of two close numbers. The nature of the error lies in the fact that the calculation is carrying out with the same relative precision. Thus the truncation error increases in a continuous reduction of the step size *h* caused by the difference between two close numbers to and after that becomes dominant over the truncation error. Hence, for the effective use of the finite-difference method is required to find a compromise between these two errors in selecting the step size *h*. On the other hand, there may be other approximation of derivatives on the basis of the finite difference method using the Taylor expansion to improve the situation in terms of the mentioned errors but they do not remove until the end of the problem and ensure the accuracy of the step size selection. The derivative value for the central differences is determined as follows:

$$\frac{df(x)}{dx} = \frac{f(x+h) - f(x-h)}{2h} + O\left(h^2\right). \qquad (2)$$

In the equation (2) the truncation error is not less than h^2. Therefore we can improve the relationship between the two errors and thereby improve the accuracy of the calculation of derivatives.

There is another way to improve the accuracy of the calculation of the derivatives using the complex step method. It is possible or even get rid of the above errors or to reduce their impact on each other using non real number. The complex step method yields the following expression for the derivative:

$$\frac{df(x)}{dx} = \frac{\mathrm{Im}(f(x+ih))}{h} + O\left(h^2\right) . \qquad (3)$$

Equation (3) can be obtained by expansion of a neighborhood of a point *x* in a Taylor series giving increment its complex argument:

$$\left.\begin{aligned} f(x+ih) = f(x) + ih\frac{f'(x)}{1!} + \ldots \\ \ldots - h^2\frac{f''(x)}{2!} - ih^3\frac{f'''(x)}{3!} + \ldots \end{aligned}\right\}, \qquad (4)$$

and taking the imaginary part of the equation (4) drive at the desired equation (3). Truncation error using the complex step method is the same order as for the central differences. However, unlike the finite difference methods we can achieve a significant reduction in its value by setting the small arbitrary step. At the same time due to the lack of subtraction in the equation (3), the rounding error in the difference between two close numbers almost does not manifest itself for any functions. Thus, putting the small step size we can achieve a relative accuracy of the computation of the derivative which equal to the accuracy of calculation of the function *f(x)*.

Fig. 1 shows the comparison of the relative error of the derivative calculation of *tan (x)* by using the central difference method and the complex step method with the step size *h*. It is seen that decreasing the step size relative error in computing the derivative using the central difference (blue line in the figure) decreases monotonically at first, as long as the prevalent component of the truncation error while achieving a minimum. Then begin to dominate component of the rounding error when the difference between two close numbers which leads to a monotonic increase in the relative error in the determination of the derivative with a further reduction of the step size. The relative error when using the complex step method (red line) decreases monotonically with the step size reaching a value corresponding to the relative accuracy of the calculations.

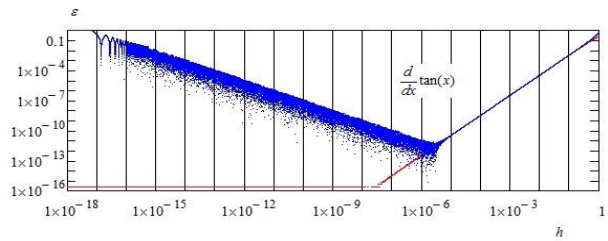


Fig.1. the comparison of the relative error of the derivative calculation of *tan (x)* by using the central difference method and the complex step method with the step size *h*

However, unfortunately, not all functions can achieve the same picture by using the complex step method. Here we are again faced with the manifestation of rounding errors. This is due to the fact that by reducing the step size *h*, imaginary parts of equation (3) become very small. A process of calculating some of the functions in the complex domain itself may have a differential representation. This can be illustrated by calculating the derivative of *arctg(x)*. During the calculation, this function has the following representation: $arctg(x+ih)=-(i/2)\ln[(1+ix-h)/(1-ix+h)]$. Manifestation of rounding errors in this case can be illustrated in Figure 2, where the blue line corresponds to the using the central difference method and the red line corresponds to the using the complex step method. It can be seen in the figure that the behavior of the relative error as a step size of the function in this case is the same for complex step method and central difference method. The use of central differences even allows to achieve the better result.

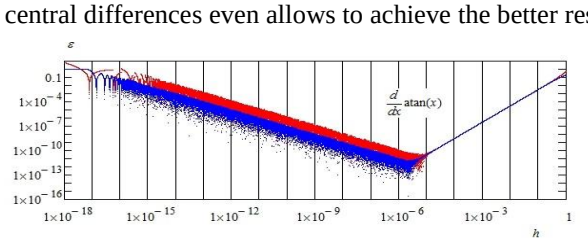


Fig.2. the comparison of the relative error of the derivative calculation for *atan (x)* by using the central difference method and the complex step method with the step size *h*

In addition, it should be noted that the computation of certain functions in the complex domain as well as a

number of computational operations is very costly in terms of computing resources. A small order of the imaginary parts of the complex number using the complex step method reduces the efficiency of its practical application, for example, to calculate the sensitivity of the matrix elements in the solution of nonlinear boundary value problem by continuation method.

The dual numbers are presented in detail in [3]. We note only that the ideal is the number of the form $<x,x'>=x+dx'$, where x, $x'\in\mathbf{R}$, x - the real part, x' - dual part, $d^2=0$. For dual numbers with a single dual part of an isomorphism with the space $\mathbf{R}^2$ is analogy with complex step method. The dual number with a vector of the dual part - number of the form $<x,x',x'',..>=x+x'd_1+x''d_2+\ldots$; where $x,x',x'',\ldots\in\mathbf{R}$, x - real part, $x',x'',\ldots$ - make up part of the dual vector, $d_1^2= d_2^2=\ldots d_n^2=0$ и $d_id_j=0$ while $i\neq j$.

Method of calculation of derivatives is presented in [3]. We give it for the case of operations on real numbers with a single dual part. We represent the function $f(x)$ in a Taylor series in the neighborhood giving it a purely point x dual increment:

$$\left.\begin{aligned} f(x+dh) &= f(x)+dh\frac{f'(x)}{1!}+\ldots \\ \ldots &+(dh)^2\frac{f''(x)}{2!}+(dh)^3\frac{f'''(x)}{3!}+\ldots \end{aligned}\right\}. \qquad (5)$$

Since in the equation (5) all the terms for which dh has power greater than two are set to zero and the definition of the dual number expression is as follows:

$$f(x+dh)=f(x)+dh\,f'(x). \qquad (6)$$

Setting this value at the real part h=1, we obtain according to (6) the following expression for the derivative:

$$\frac{df(x)}{dx}=Dp\big(f(x+1d)\big). \qquad (7)$$

Of course the value of the function $f(x)$ corresponds to the real part of the dual number. To obtain the required value of the derivative of the function calculating its value in the dual representation is enough and to take away from the dual part. In this case, the implementation of the computational scheme (generating the necessary calculations in dual numbers) automatically obtains the corresponding derivatives.

Since the equation (6) , the truncation error in the calculation of the derivatives by dual numbers is equal to zero. The equation (7) is similar to the of complex step method which describe in [4]. But the difference is the actual orders and the dual part of the right side of the equation (7) or the same as or may always be selected as such an appropriate step size h. Therefore, we do not face the possible rounding error in the difference between the two close numbers in the calculation of the dual representation $f(x)$ unlike the use of complex steps method. It is possible in the process of computation automatically to obtain the required values of derivatives with relative precision which equal to the precision of computing functions.

In the case of calculating the partial derivatives of functions of several variables similarly applied to the dual vector of the dual part [3]. The dimension of the dual vector is determined by the number of function arguments. Thus, for the function of two variables, we can use the following relations:

$$\left.\begin{aligned} &y=f(x_1,x_2),\ x_1+x_1'd_1+0d_2, \\ &x_2+0d_1+x_2''d_2, \\ &y=<y,y',y''>=y+y'd_1+y''d_2, \\ &x_1'=x_2''=1, \\ &f(x_1,x_2)=Rp(y), \\ &\frac{\partial f}{\partial x_i}=Dp_i(y),\ i=1,2. \end{aligned}\right\} \qquad (8)$$

Thus the computation of the function of several variables on the relations analogous to (8), together with the value of the function, we automatically receive the derivatives values of all its arguments with the same accuracy. The above specified advantages of using dual numbers method in comparison with the complex step method and the much smaller amount of computation required in the case of the same operations are indicated in [3].

The dual numbers is used for automatic differentiation as vector part of the dual numbers and for the unit numbers. The dual numbers with a single part are encouraged to use in solving the nonlinear boundary value problem of the maximum principle by continuation method to compute the sensitivity matrix. It is dedicated to the next section of the article. The dual part of the vector is proposed to use in solving the optimization problem for the multi-revolutions orbital transfer taking into account the most complete perturbation model. In this case it is assumed that the automatic calculation of the right sides of the optimal motion equations of the spacecraft by calculating the optimal Hamiltonian (by virtue of the canonical formalism of the maximum principle).

3. APPLICATION OF THE DUAL NUMBERS METHOD IN CONTINUATION METHOD

The formalism of the Pontryagin maximum principle reduces search extremal optimization problems to the solution of the corresponding boundary value problem which is equivalent to the system of nonlinear transcendental equations of the following form:

$$\mathbf{f}(\mathbf{p})=0,\ \mathbf{f}:\mathbf{R}^n\to\mathbf{R}^n,\quad \mathbf{p}\in\mathbf{R}^n. \qquad (9)$$

Using continuation method [5] for the solution of the system (9) as compared to the conventional methods of the first order has a number of advantages, the main of which is its inherent global convergence in carrying out a number of assumptions [5,6].

Let consider the inner Cauchy's problem for a system of differential equations describing the optimal process with a *smoothed* piecewise continuous control:

$$\frac{d\mathbf{x}}{dt} = \varphi(\mathbf{x}, t, \tau), \mathbf{x}(t_0) = \mathbf{x}_0, t \in [t_0, t_1]. \quad (10)$$

Where $\mathbf{x}$:$t \to \mathbf{R}^N$, $\varphi(\mathbf{x},t,\tau) \in C^1(\mathbf{R}^N \times \mathbf{R} \times \mathbf{R})$. Introducing the option of continuing in the boundary conditions that define the system (5), we obtain the following representation: $\mathbf{K}(\mathbf{x}(t_0),\mathbf{x}(t_1),t_0,t_1,\tau)=0$, $\mathbf{K} \in C^1(\mathbf{R}^{2N+3})$. Thus the system (5) is now explicitly dependent on the continuation on the parameter and represented in the form $\mathbf{f}(\mathbf{p},\tau)=0$. Then the external Cauchy's problem [5,6] is formulated as follows:

$$\left.\begin{aligned} &\frac{d\mathbf{p}}{d\tau} = -\left(\left.\frac{\partial \mathbf{f}}{\partial \mathbf{p}}\right|_{\mathbf{p}=\mathbf{p}(\tau)}\right)^{-1}\left(\mathbf{f}(\mathbf{p}_0) + \frac{\partial \mathbf{f}}{\partial \tau}\right), \\ &\mathbf{p}(0) = \mathbf{p}_0, \quad \tau \in [0,1]. \end{aligned}\right\} \quad (11)$$

Integrating external Cauchy's problem reaches to the solution of the system (9). This approach has been used successfully in [5,6]. In particular, it enables to continuous homotopy between the auxiliary (simpler) and the main task - relevant system (9); it is assumed that the solution of auxiliary system is known.

It is obvious that the accuracy of the calculation of the elements of sensitivity matrix for the system (11), a member of the right-hand sides of the differential equation of the Cauchy problem directly depends on the convergence of the continuation method. When the numerical approximation of the partial derivatives of (9) for the unknown parameters of the boundary value problem is calculated, the error in determining the derivative is accumulated in external integration which ultimately could lead to the gathering on the continuation path. It also has a negative impact on the process of external integration - the noise in the right-hand sides of differential equations of the Cauchy problem leads to a reduction in the integration step when the automatic choice and the overall reduction in the stability of the computational process. Therefore, in view of the above reasons, in this paper we propose to use the dual numbers method to calculate the partial derivatives constituting the sensitivity matrix of the system (9).

It is supposed to build the computational scheme using dual numbers method with a single dual part. The advantages of the dual numbers compared to the complex step method and finite difference methods have been stipulated earlier. At one step of integration of the external relative error in the calculation of the partial derivatives will not be greater than the given relative accuracy of internal integration. By increasing the accuracy of external integration it is possible to reduce the error of initial data (at each step of the external integration) for internal integration, As a result, the total error in calculating the sensitivity matrix is reduced. By increasing the accuracy of both internal and external integration, it is possible to achieve a more sustainable to continue the process with a significantly smaller number of discarded steps than in the case of using the complex step method to calculate partial derivatives.

This is due to the fact that when using the complex step method to reduce the truncation error we should select the step size as small as possible. The consequently, during the computation in the imaginary part of the complex domain of the expressions, there is the order of increment (or less). Therefore, calculations in the complex region will manifest the rounding error due to the difference between the two close numbers.

It is obvious that the calculation in elements of the sensitivity matrix is required to carry out internal integration over the field of dual numbers. However, given the complexity of the numerical realization of such integration, it is advisable to use the following (equivalent to it) calculation scheme:

$$\left.\begin{aligned} &\mathbf{y} \in \mathbf{R}^{2N}, \mathbf{x}^d \in DualNumber^N \\ &y_i = \begin{cases} Rp(\varphi(\mathbf{x}^d)_i), i = 1,2..N \\ Dp(\varphi(\mathbf{x}^d)_{i-N}), i = (N+1)..2N \end{cases} \\ &\dot{\mathbf{y}} = \mathbf{y}, \to \mathbf{y1} \end{aligned}\right\} \quad (12)$$

According to (12), the new expanded system of differential equations y of order 2N is considered in the real domain. We calculate the right hand side of the system of differential equations $\varphi(\mathbf{x}^d)$ (order N) for the inner Cauchy's problem in the dual representation. Right hand side of the expansion of the system $\dot{\mathbf{y}}$ are assigned to the respective real and dual parts of the value of the previously calculated the dual representation of right hand sides of differential equations; the values of the extended state vector **y1** of the system are determined during the next step of integration. Further, based on the expanded phase vector **y1** of the system newly formed the dual representation of the phase vector of the system - $\mathbf{x}^d$.

The above scheme is repeated at each step of the internal integration. Integrating the internal Cauchy problem and calculating the residual dual representation on the boundary conditions, together with their values at the same time we get and we need to calculate the derivatives of the sensitivity matrix.

4. THE PROCEDURE FOR CALCULATING OF THE RIGHT HAND SIDE OF OPTIMAL MOTION EQUATIONS BY DUAL NUMBERS METHOD

This procedure is based on the use of the canonical formalism of the maximum principle: optimum Hamiltonian completely defines the system, describing the optimal process. It takes only one calculation of optimal Hamiltonian in the dual representation for each access to the right parts of the numerical integration of the system of motion equations of the spacecraft. The procedure can be described by the following scheme:

where - $\mathbf{x}$, $\boldsymbol{\psi}$ - phase vector and adjoints vector of the system (dual representation) respectively; x_i, ψ_i - components of these vectors; $x_{i,0}$, $\psi_{i,0}$ - the real part of the phase component and the adjoints vectors; $H^0(\boldsymbol{\psi},\mathbf{x})$ - dual representation of the of the optimal Hamiltonian; $\dot{x}_i$, $\dot{\psi}_i$ - the actual values of the right sides of the corresponding differential equations; H^0 - the actual value of the optimal Hamiltonian.

$$\left.\begin{aligned}
&i=1,2...n,\ j=1,2...2\cdot n,\\
&\mathbf{x}=\left(x_1,..,x_i,..x_n\right)^T,\\
&\boldsymbol{\psi}=\left(\psi_1,..,\psi_i,..\psi_n\right)^T,\\
&x_i=x_{i,0}+\sum_{j=1}^{2n}x_{i,j}\cdot d_j,\\
&\psi_i=\psi_{i,0}+\sum_{j=1}^{2n}\psi_{i,j}\cdot d_j,\\
&\left.\begin{aligned}x_{i,j}&=1,\ j=i\\ x_{i,j}&=0,\ j\neq i\end{aligned}\right\},\quad \left.\begin{aligned}\psi_{i,j}&=1,\ j=i+n\\ x_{i,j}&=0,\ j\neq i+n\end{aligned}\right\},\\
&\dot{x}_i=Dp_{n+i}(H^0(\boldsymbol{\psi},\mathbf{x})),\\
&\dot{\psi}_i=-Dp_i(H^0(\boldsymbol{\psi},\mathbf{x})),\\
&H^0=Rp(H^0(\boldsymbol{\psi},\mathbf{x})).
\end{aligned}\right\}$$

5. MATHEMATICAL MODEL OF THE SPACECRAFT'S MOTION

The motion of the SC with EP is described by the following system of differential equations in the equinoctial elements [7]:

$$\left.\begin{aligned}
&\frac{dp}{dt}=2\,\frac{p^{3/2}}{b_1}T\\
&\frac{de_x}{dt}=\frac{p^{1/2}}{b_1}\begin{pmatrix}\left(\left(1+b_1\right)\cos(l)+e_x\right)T\\ +b_1\sin(l)\ S-e_y\ b_2\,W\end{pmatrix}\\
&\frac{de_y}{dt}=\frac{p^{1/2}}{b_1}\begin{pmatrix}\left(\left(1+b_1\right)\sin(l)+e_y\right)T\\ -b_1\cos(l)\ S+e_x\ b_2\ W\end{pmatrix}\\
&\frac{di_x}{dt}=\frac{1}{2}\frac{p^{1/2}}{b_1}b_3\cos(l)\,W\\
&\frac{di_y}{dt}=\frac{1}{2}\frac{p^{1/2}}{b_1}b_3\sin(l)\,W\\
&\frac{dl}{dt}=\frac{b_1^{\ 2}}{p^{3/2}}+p^{1/2}\,\frac{b_2}{b_1}W,
\end{aligned}\right\}\qquad(13)$$

where $b_1=1+e_x\cos(l)+e_y\sin(l)$, $b_2=i_x\sin(l)-i_y\cos(l)$, $b_3=1+i_x^2+i_y^2$. Equinoctial elements p,e_x,e_y,i_x,i_y,l are defined by classical orbital elements: $e_x=e\cos(\Omega+\omega)$, $e_y=e\sin(\Omega+\omega)$, $i_x=\tan(i/2)\cos(\Omega)$, $i_y=\tan(i/2)\sin(\Omega)$, $l=\Omega+\omega+\upsilon$, where p - semi-latus rectum, e - eccentricity, i - inclination, Ω - right ascending node, ω - pericenter argument, υ - true anomaly , l - true longitude. S, T, W – reactive components and disturbing accelerations acting on the spacecraft:

$$\left.\begin{aligned}
S&=S_a+\sum_j S_{\Phi j},\\
T&=T_a+\sum_j T_{\Phi j},\\
W&=W_a+\sum_j W_{\Phi j}.
\end{aligned}\right\}$$

Where $S_{\Phi j}$, $T_{\Phi j}$, $W_{\Phi j}$ - acceleration components of the spacecraft acquired them under the influence of perturbing forces of different nature; S_a, T_a, W_a - components of the reactive acceleration which are determined depending on the model of the electric propulsion system. Thus for the model of unregulated limited thrust they are defined as follows:

$$\left.\begin{aligned}
S_a&=\delta\left(P/m\right)\sin(\vartheta)\cos(\psi),\\
T_a&=\delta\left(P/m\right)\cos(\vartheta)\cos(\psi),\\
W_a&=\delta\left(P/m\right)\sin(\psi).
\end{aligned}\right\}$$

Where P - Thrust, m - mass of SC; θ, ψ -pitch and yaw angles, δ - indicator of the engine, $\delta\in\{0,1\}$. Three functions $\theta(t)\in[0,2\pi)$, $\psi(t)\in[-\pi/2,\ \pi/2]$, $\delta\in[0,1]$ define the control vector of the spacecraft $\mathbf{u}_{OT}=(\theta\ ,\psi,\ \delta)$. For the model of power-limited engine the thrust acceleration components S_a, T_a, W_a are defined by the following: $S_a=a_S$, $T_a=a_T$, $W_a=a_W$. Where the control vector $\mathbf{u}_{OM}=(a_S,\ a_T,\ a_W)$ is defined by the three functions $a_S(t)$, $a_T(t)$, $a_W(t)$.

In the case of the consideration of spacecraft motion with limited engine thrust system (13) is complemented by the equation describing the change of its mass over time: $dm/dt=-\delta(P/w)$, where w - effective exhaust velocity of EP. For power-limited case the equation becomes: $dm/dt=-(m^2a^2)/(2N_r)$, $a=\|a_S,a_T,\ a_W\|$, where N_r - reactive power of the electric propulsion systems.

6. FORMULATION OF OPTIMIZATION PROBLEM

It is required an optimum transfer in terms of the required amount of fuel between the initial and final orbits in a fixed time. Thus as the objective, we consider the following:

$$J=-m\left(t_f\right)\to\min,\qquad(14)$$

meets the minimum fuel consumption.

Each of the two models under consideration for the electric propulsion systems gives the separate optimization problem. We call the problem (14) for the model of power-limited engine as OM problem and OT problem for non-limited engine thrust following the terms of the author which are introduced in [5,6]. The formalism of the maximum principle is used [1,2] to solve this problem (14). The Hamilton-Pontryagin function as follows:

where H_1^{OT} and H_1^{OM} - of the Hamilton-Pontryagin function, depending on the control for OT and OM problem respectively; $\boldsymbol{\psi}=(\psi_p,\psi_{ex},\psi_{ey},\psi_{ix},\psi_{iy},\psi_l),\psi_m$ - adjoints variables. Terminant of the problem: $\ell=\alpha_0(-m(t_f))+\boldsymbol{\beta}_0\mathbf{f}_0(\mathbf{x}(t_0))+\boldsymbol{\beta}_k\mathbf{f}_k(\mathbf{x}(t_f))+\beta_{m0}(m(t_0)-m_0)$, where $\mathbf{f}_0(\mathbf{x}(t_0))$, $\mathbf{f}_k(\mathbf{x}(t_f))$ - vector function which define the parameters of

initial and final orbits $\mathbf{x}=(p,e_x,e_y,i_x,i_y,l)$; α_0, β_{m0} and vectors $\boldsymbol{\beta}_0$, $\boldsymbol{\beta}_k$ - Lagrange multipliers, m_0 - initial mass of the SC.

$$\left.\begin{aligned}
&H = H_1 + H_2 + H_3\\
&H_1^{OT} = \delta\left(P/m\right)\begin{pmatrix} a_1\sin(\vartheta)\cos(\psi)+\ldots\\ \ldots + a_2\cos(\vartheta)\cos(\psi)+\ldots\\ \ldots + a_3\sin(\psi)\end{pmatrix}+\ldots\\
&\ldots.-\psi_m\delta\left(P/w\right)\\
&H_1^{OM} = a_S a_1 + a_T a_2 + a_W a_3 + \ldots\\
&\ldots.-\psi_m\frac{m^2\left(a_S^{\,2}+a_T^{\,2}+a_W^{\,2}\right)}{2N_r}\\
&H_2 = \psi_l\frac{b_1^{\,2}}{p^{3/2}},\\
&H_3 = a_1\sum_j S_{\Phi j} + a_2\sum_j T_{\Phi j} + a_3\sum_j W_{\Phi j},\\
&a_1 = p^{1/2}\left(\psi_{e_x}\sin(l)-\psi_{e_y}\cos(l)\right)\\
&a_2 = \frac{p^{1/2}}{b_1}\begin{pmatrix} 2\,p\,\psi_p + \psi_{e_x}\left((1+b_1)\cos(l)+e_x\right)+\ldots\\ \ldots+\psi_{e_y}\left((1+b_1)\sin(l)+e_y\right)\end{pmatrix}\\
&a_3 = \frac{p^{1/2}}{b_1}\begin{pmatrix} b_2\left(\psi_{e_y}e_x-\psi_{e_x}e_y+\psi_l\right)+\ldots\\ \ldots+\frac{b_3}{2}\left(\psi_{i_x}\cos(l)+\psi_{i_y}\sin(l)\right)\end{pmatrix}
\end{aligned}\right\} \quad (15)$$

Optimal control in case of OT is defined as follows:

$$\begin{pmatrix} u_1^{\,0}\\ u_2^{\,0}\\ u_3^{\,0}\\ u_4^{\,0}\end{pmatrix} = \begin{cases} a_2\Big/\sqrt{a_1^{\,2}+a_2^{\,2}};\\ a_1\Big/\sqrt{a_1^{\,2}+a_2^{\,2}};\\ \sqrt{a_1^{\,2}+a_2^{\,2}}\Big/\sqrt{a_1^{\,2}+a_2^{\,2}+a_3^{\,2}};\\ a_3\Big/\sqrt{a_1^{\,2}+a_2^{\,2}+a_3^{\,2}}.\end{cases}$$

$$\delta^0 = \begin{cases} 1, \quad \Pi > 0;\\ 0, \quad \Pi < 0;\\ \forall\delta\in(0\,,1), \quad \Pi = 0.\end{cases} \quad (16)$$

where $u_1^{\,0}=\cos(\theta)$, $u_2^{\,0}=\sin(\theta)$, $u_3^{\,0}=\cos(\psi)$, $u_4^{\,0}=\sin(\psi)$, $\Pi = \frac{P}{m}\sqrt{a_1^{\,2}+a_2^{\,2}+a_3^{\,2}} - \psi_m\frac{P}{w}$ - switching function.

Optimal control in case of OM is the followings:

$$a_S^{\,0} = \frac{N_r}{\psi_m m^2}a_1,\; a_T^{\,0} = \frac{N_r}{\psi_m m^2}a_2,\; a_W^{\,0} = \frac{N_r}{\psi_m m^2}a_3. \quad (17)$$

The equations of the adjoints system for OT and OM problems with the expressions (15), (16) and (17) are defined as follows:

$$\left.\begin{aligned}
&\frac{d\boldsymbol{\psi}}{dt} = -\frac{\partial H}{\partial\mathbf{x}}\\
&\frac{d\psi_m}{dt} = \delta\frac{P}{m^2}\sqrt{a_1^{\,2}+a_2^{\,2}+a_3^{\,2}},
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&\frac{d\boldsymbol{\psi}}{dt} = -\frac{\partial H}{\partial\mathbf{x}}\\
&\frac{d\psi_m}{dt} = \frac{N_r}{\psi_m m^3}\left(a_1^{\,2}+a_2^{\,2}+a_3^{\,2}\right).
\end{aligned}\right\}$$

Transversality conditions for OT and OM problems:

$$\boldsymbol{\psi}(t_0) = \left.\frac{\partial\ell}{\partial\mathbf{x}}\right|_{t_0},\; \boldsymbol{\psi}(t_k) = -\left.\frac{\partial\ell}{\partial\mathbf{x}}\right|_{t_k},$$

$$\psi_m(t_0) = \left.\frac{\partial\ell}{\partial m}\right|_{t_0},\; \psi_m(t_k) = -\left.\frac{\partial\ell}{\partial m}\right|_{t_k}$$

The next condition is a consequence of the transversality conditions and the non-negativity, and is performed for both problems: $\psi_m(t_f)=\alpha_0$, $\alpha_0\geq 0$. Selecting α_0 determines the appropriate normalization of Lagrange multipliers. In the future, we consider $\alpha_0=1$.

Optimal Hamiltonian for OT problem is defined as follows:

$$\begin{aligned} H^{OT} &= \delta\left(P/m\right)\sqrt{a_1^{\,2}+a_2^{\,2}+a_3^{\,2}}+\ldots\\ &\ldots-\psi_m\delta\left(P/w\right)+H_2+H_3\end{aligned}, \quad (18)$$

and for OM problem is defined as follows:

$$H^{OM} = \frac{N_r}{2\psi_m m^2}\left(a_1^{\,2}+a_2^{\,2}+a_3^{\,2}\right)+H_2+H_3. \quad (19)$$

Equations (18) and (19) are proposed to use for the automatic calculation of the right hand sides of the system of differential equations of optimal motion of spacecraft, taking into account the existing the perturbations of different nature. In this case, the nonlinear boundary value problem of the maximum principle (9) (for OM and OT-problem) is solved by using hybrid method of Powell.

To solve the OT problem excluding perturbations acting on the spacecraft will use the methodology which is described in [5,6], based on the application of the continuation method. To solve this problem we use the homotopy between OM and OT providing the continuous and smooth extension of the solution of OM problem to the solution of OT problem. This is achieved by introducing into the right hand sides of the system of differential equations of optimal motion of the spacecraft continued on parameter. The system of differential equations of optimal motion of spacecraft providing a continuous extension of the OM to OT problem is as follows:

$$
\left.
\begin{aligned}
&\mathbf{x} = \left(p, e_x, e_y, i_x, i_y, l\right)^T, \\
&H^{OM} = H_1^{OM} + H_2, \\
&H^{OT} = H_1^{OT} + H_2, \\
&\dot{\mathbf{x}} = \frac{\partial H^{OM}}{\partial \boldsymbol{\psi}} + \tau\left(\frac{\partial H^{OT}}{\partial \boldsymbol{\psi}} - \frac{\partial H^{OM}}{\partial \boldsymbol{\psi}}\right), \\
&\dot{\boldsymbol{\psi}} = \tau\left(\frac{\partial H^{OM}}{\partial \mathbf{x}} - \frac{\partial H^{OT}}{\partial \mathbf{x}}\right) - \frac{\partial H^{OM}}{\partial \mathbf{x}}, \\
&\dot{\psi}_m = \tau\left(\frac{\partial H^{OM}}{\partial m} - \frac{\partial H^{OT}}{\partial m}\right) - \frac{\partial H^{OM}}{\partial m}.
\end{aligned}
\right\} \qquad (20)
$$

In this case, to ensure the continuity of the right-hand sides of the equation (20), delta function is defined as: δ=0.5(1+Π/(|Π|+ε)),, where ε - regularizator. The smaller the value of ε, the closer it gets to the function of a piecewise constant.

7. NUMERICAL RESULTS

The solutions of OT problem without the perturbations produced by the method of continuation and approximation of the sensitivity of the matrix elements of dual numbers with a single dual part are described in the following. The results of OT-problem with the perturbations in the case of dual numbers with vector dual part used for calculating the right sides of the spacecraft optimal motion system are also given.

7.1 OT problem without the perturbations

We considered the following orbital transfer problem to geostationary (GEO) orbit. The initial orbit has the following characteristics: focal parameter p = 20,000 km, eccentricity e = 0.75, inclination i = 25 degrees; longitude of the ascending node and the argument pericenter are equal to zero. The initial value of the true anomaly in the orbit is fixed at 200 degrees. The angular distance of the flight is not fixed. The duration of the flight is 90 days. The characteristics of SC are as follows: initial mass of SC is equal to 1320 kg, the thrust is equal to 0.4 N, the specific impulse is equal to 1500 s. As a result of the flight, the relative final mass of SC is equal to 0.88865, the final mass of the SC is equal to 1173.18 kg. The angular distance of the flight is equal to 73.2 revolutions

Formalism of the continuation method was used in the analysis of the trajectory in terms of determining the optimum angular distance on the trajectory between these orbits. The resulting solution of the problem has been continued by the angular distance from the point corresponding to the final value of the true longitude l_k^0 of the given solution with two revolutions in the backward direction and seven revolutions in the forward direction. This has been implemented as follows: the condition $\psi_l(t_f)$=0 was used in the boundary conditions of the problem instead of the condition $l(t_f)$-($l_k^0 \pm \tau(2\pi N)$), where N - number of whole revolutions, τ - parameter of the continuation. The modified Newton homotopy is used for continuation on the boundary conditions. Each step of the method of continuation of solution of the problem corresponds to the current value of the angular distance of $l_k(\tau)=l_k^0 \pm \tau(2\pi N)$. The results for the continuation of angular distance are presented in Figure 3.

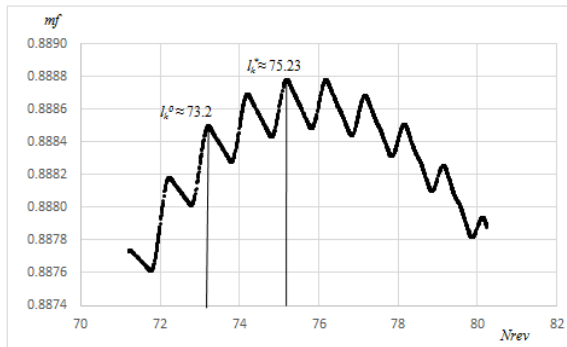


Fig.3. the final mass of the SC [dimensionless] to the total angular distance N [the number of revolutions]. The figure shows: the angular distance l_k^0, corresponding to the original solution; globally optimal value of the angular distance - l_k^*.

From Figure 3 it is cleared that the functional dependence of the total angular distance of the flights is monotonic and has among the many locally optimal solutions within each revolution, one a pronounced as the global maximum. The maximum is achieved by increasing the angular distance in two revolutions. Its optimal value was 75.23 revolutions. The value of the functional at the time is nearly equal to 0.8889. In the figure: l_k^0 is the angular distance, corresponding to the original solution; l_k^* is the global optimal value of the angular distance.

It was also implemented the solution on the continuation of the boundary conditions of the problem, in order to establish the quality of the functional dependence on the magnitude of the inclination of the initial orbit and its pericenter radius. We considered a reduction in the inclination of the initial orbit to the zero and the increasing the pericenter radius of the initial orbit to the radius of the GEO. The angular distance of the flight in both cases was not fixed. Figures 4-5 show the dependences of the dimensionless mass of the SC on the inclination and the pericenter radius of the initial orbit.

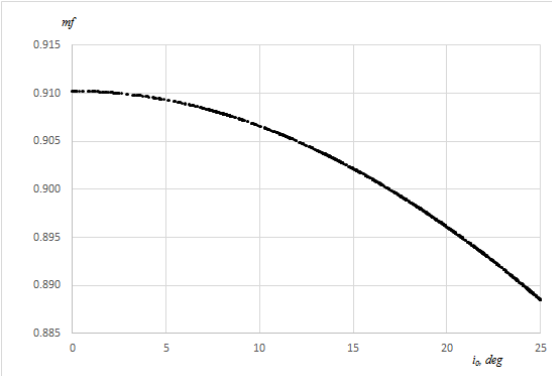


Fig.4. the final mass of the spacecraft [dimensionless] on the inclination of the initial orbit i0 [deg.].

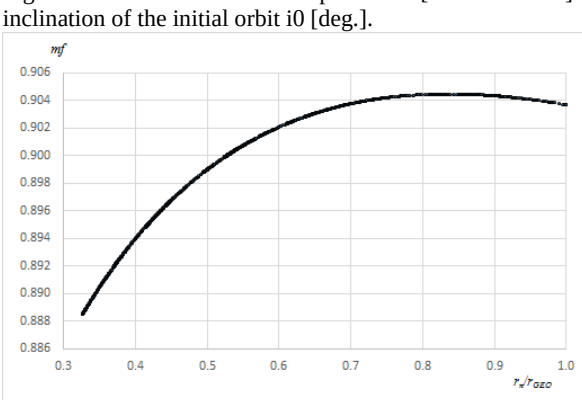


Fig.5. the final mass of the spacecraft [dimensionless] on the initial orbit pericenter radius $r_{\pi 0}$ [dimensionless], Referred to the radius of the GEO.

As seen from Fig.4 the final mass of the SC increases monotonically with decreasing initial orbit inclination, obviously thus reaching its maximum at zero.

Functional dependence of the dimensionless initial orbit pericenter radius presented in Figure 5. It increases monotonically. So, with the increase of the initial apocenter radius final mass of the spacecraft increases reaching its maximum at the pericenter radius smaller than the radius of the GEO, and then begins to decrease.

7.2 OT problem with perturbations

The perturbations acting on the spacecraft are taking into account the following: from the moon and Sun, the non-centrality of the gravitational field of the Earth. When calculating the lunisolar perturbations, coordinates of the Moon and the Sun were determined by using DE405 ephemeris supported in [8]. Centrifugal accelerations of the gravitational field of the non-centrality are calculated as derivatives of the geopotential represented as an expansion in terms of spherical functions, associated with the Earth ITRS coordinate system [9]. At the same time we take into account the precession and nutation of the Earth's axis and the movement of the pole according to the model IAU 2006 / 2000A. The necessary calculations were performed using the SOFA IAU programs [9]. To calculate the geopotential model was used Earth's gravitational field EGM-96.

Parameters of the initial orbit of the spacecraft are following: perigee altitude - 5000 km, apogee altitude - 80000 km, the inclination - 25 degrees. argument of pericenter and longitude of the ascending node assumed to be zero. True anomaly of the initial orbit - 200 degrees. The initial mass of the SC - 2500 kg thrust - 0.56 N, the specific impulse - 1640s. The duration of the flight to GEO - 120 days. We considered the geopotential 4×4. Date of start corresponds to December 26, 2015 1:00 32 minutes UTC.

The solution to this problem was obtained using the following stages using the preliminary solutions: first averaged OM problem without perturbations is solved, then averaged OM problem within the above model, after that it defines a solution of non averaged OM problem with perturbations and finally this solution was used as an initial approximation to the considered OT problem.

As a result, the dimensionless mass of SC was found as 0.89493 which is slightly less compared to the same problem without taking into account the perturbations - 0.89540.

Figures 6-11 show the main characteristics of the solution for OT problem in the given perturbations. In each figure the comparison to the given appropriate unperturbed solution is shown. In all the figures the solid line corresponds to the perturbed solution and point corresponds to unperturbed.

Figure 6 show the dependence of the semi-major axis on flight time. This fragment is enlarged depending on the range of 18 to 20 days is shown in 6b. In both cases, the semi-major axis varies non-monotonically; in this case clearly shows the difference of the two dependencies, reaches a maximum at flight time of 50 day. On fig.6b visible manifestation of disturbances acting on the non-centrality of the gravitational field of the Earth is passive the perturbed trajectories of spacecraft while passing periapsis.

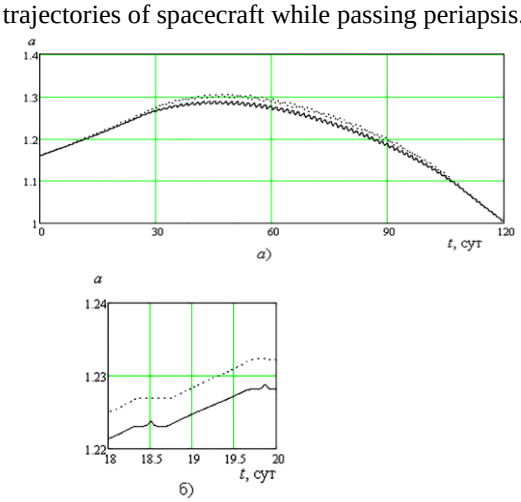


Fig.6. The semimajor axis of the orbit of the spacecraft (a) for the entire duration of the flight; b) in the range from 18 to 20 days.

Figure 7 and 8 show the dependences of longitude of the ascending node and argument pericenter on the flight time. It can be seen clearly the contribution of the non-

centrality of the gravitational field of the Earth in the evolution of these elements.

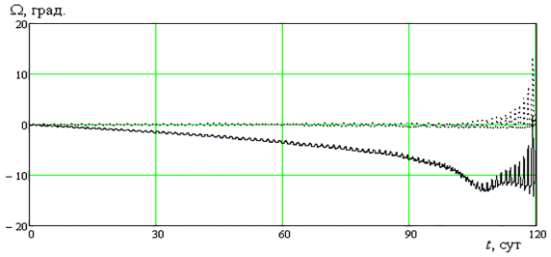


Fig. 7. The longitude of the ascending node of the spacecraft on flight time

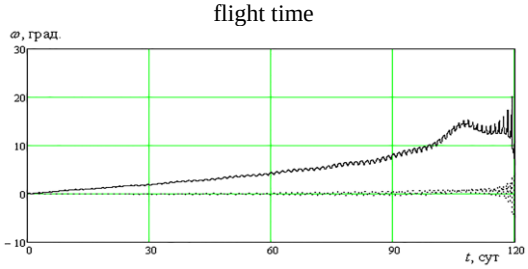
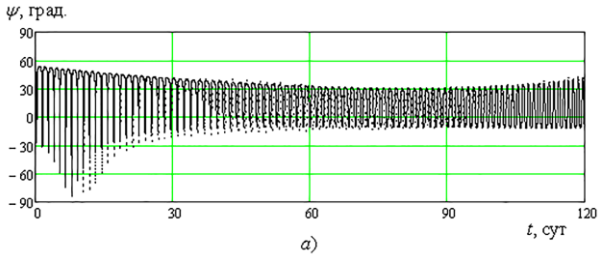
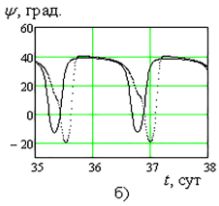


Fig.8. The argument of pericenter of the spacecraft on flight time

Figure 9a and 10a show the obtained optimal spacecraft control program for yaw and pitch angles. Enlarged fragments of these dependencies are shown in figures 9b and 10b. It can be seen that the change in the yaw and pitch angles for the perturbed motion ahead of the corresponding change in phase to the unperturbed; thus, the nature of dependencies generally repeated. The switching functions of the electric propulsion systems are similar. Fragments of these dependences are presented in Figure 11.

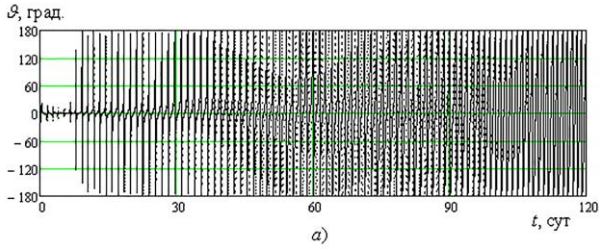


а)

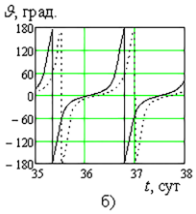


б)

Fig.9. The yaw angle: a) for the entire flight time; b) in the range from 35 to 38 days.



а)



б)

Fig.10. The pitch angle: a) for the entire flight time; b) in the range from 35 to 38 days.

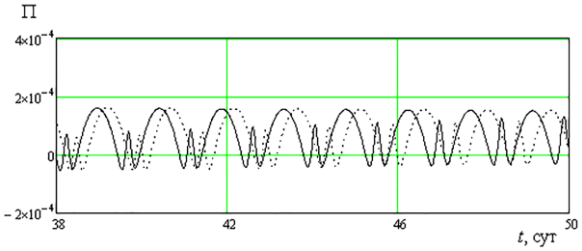


Fig.11. Switching functions of the electric propulsion system on spacecraft flight duration from 38 to 50 days

8. CONCLUSION

This paper analyzes two possible aspects of using the dual numbers method for solving optimization problems for the spacecraft orbital transfer with EP. The first of these corresponds to the use of dual numbers with a single dual part to calculate the elements of sensitivity matrix in the solution of nonlinear boundary value problems of the maximum principle by the continuation method. The second aspect involves the use of dual numbers method with the vector of the dual parts for the purpose of calculating the right hand sides of the optimal motion of the system in the case of solving the problem of optimizing the spacecraft orbital transfer, taking into account the most complete model of perturbations. The results using the dual numbers method for the fuel optimal transfer problem with EP from initial elliptical orbit into geostationary orbit are presented and analyzed.

9. REFERENCES

[1] Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V., Mishchenko E.F., *"Matematicheskaya teoriya optimal'nykh protsessov* (The mathematical theory of optimal processes)", Nauka, Moscow, 1969.

[2] Afanas'ev A.P., Dikusar V.V., Milyutin A.A., Chukanov S.A., "*Neobkhodimoe uslovie v optimal'nom upravlenii* (The necessary condition in the optimal control)", Nauka, Moscow, 1990.

[3] Yu W., Blaire M., "DNAD A Simple Tool for Automatic Differentiation of Fortran Codes Using Dual Numbers", *Computer Physics Communications*, vol.184, no. 5, pp. 1446-1452, 2013

[4] Martins J. R. R. A., Sturdza P., Alonso J.J., "The Complex-Step Derivative Approximation", *ACM*

Transactions on Mathematical Software, Vol. 29, no. 3, pp. 245–262., september 2003
[5] Petukhov V.G. “Metod prodolzheniya dlya optimizatsii mezhplanetnykh traektorii s maloi tyagoi” *Kosmicheskie issledovaniya*, vol. 50, no.3, pp. 258 – 270, 2012
[6] Petukhov V.G. *Optimizatsiya traektorii kosmicheskikh apparatov s elektroraketnymi dvigatel'nymi ustanovkami metodom prodolzheniya* (The optimization of the spacecraft trajectories with electric propulsion system by continuation method), Doctor’s thesis, Moscow, MAI, 2013
[7] Brouke R.A., Cefola P.J. “On the Equinoctial Orbital Elements”, *Celestial Mechanics*, vol.5, pp. 303-310, 1972
[8] Standish E.M., Newhall X.X., Williams J.G., Folkner W.F., “JPL Planetary and Lunar Ephemerides, DE403/LE403”, JPL IOM 314.10-127, pp.1-27,1995
[9] The SOFA Software Libraries. International Astronomical Union. Division 1: Fundamental Astronomy, Commission 19: Rotation of the Earth, Standards Of Fundamental Astronomy Board, available at www.iausofa.org, 2013