A fast and efficient algorithm for onboard LEO intermediary propagation

Denis Hautesserres‡ and Martin Lara†

(‡) Centres de Competence Tech., CNES, Toulouse, France
(†) Space Dynamics Group, UPM, Madrid, Spain & GRUCACI – Univ. La Rioja, Logroño, Spain

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Background: Onboard propagation

- Common onboard orbit propagators
  - provide satellite navigation solution for short time spans
  - for instance: in case of GPS outage (minutes . . . )
- Accumulation of high-order effects is insignificant
  - main perturbation affecting operational LEO sats. is $J_2$
  - motion in this potential field known as main problem
- Hardware and software improvements:
  - numerical solution very fast and accurate
- Analytical approach: no need of integrating ODEs
  - “instant” but approximate solution
- Main problem is non-integrable $\rightarrow$ intermediaries
• Intermediaries: **integrable** problems
  – comprise the bulk of the original main problem dynamics
  – solution limited in precision to 1st order effects of $J_2$
    * 7-8 meters in the ii.cc. of LEO
  – intermediary competitive vs. Runge-Kutta integration
• Intermediary approach: no need of integration ODEs
  – less power consumption
    * important if power constraints, like Cubesats
  – more versatile: no need of step by step evaluation
  – less precise, but shares the same statistics as R-K
    * ii.cc. are known onboard within some uncertainty
Outline

• Perturbed Keplerian motion: Geopotential
• Main problem Hamiltonian
  – truncations in polar-nodal variables
  – radial and zonal intermediaries (common and natural)
• Deprit’s radial, natural intermediary
  – 2nd order improvements
  – $J_3$ and $J_4$ effects
• Examples and comparisons
• Conclusions and future work
Perturbed Keplerian motion

- \( V = -(\mu/r) [1 + \epsilon D(r, \dot{r}, t)] \), \( \epsilon \ll 1 \), \( \mu = GM_\oplus \)
  - the disturbing function \( D \) usually makes \( V \) non-integrable
  - expected solution: slightly distorted Keplerian orbit

- Geopotential: usual expansion in spherical harmonics
  - Earth case: \( J_2 = -C_{2,0} = O(10^{-3}) \), \( C_{n,m} = O(J_2^2) \)

- Main problem: \( V = -(\mu/r) \left[ 1 - J_2 \left( \frac{R_\oplus}{r} \right)^2 P_{2,0}(\sin \phi) \right] \)
  - quite representative of the real dynamics, non-integrable
  - averaged dynamics: secular effects
    * precess. ellipse, regression/advance of the perigee
    * also, modification of the mean motion
  - non-integrable: solution by numerical integration
  - alternative: approximate analytical intermediary solutions
Truncations of the main problem

- Hamiltonian formalism (≈ energy, canonical variables)
  - state scalar function from which we derive the e.o.m.
  - polar variables \( r, \theta, \nu, R = \dot{r}, \Theta = r^2 \dot{\theta}, N = \Theta \cos i \)
- Expand the Legendre polynomial: \( \sin \varphi = \sin i \sin \theta \)

\[
\mathcal{H} = \frac{1}{2}(R^2 + \Theta^2/r^2) - \frac{\mu}{r} - \frac{1}{2}(\mu/r)(R_{\oplus}/r)^2 J_2 \quad \text{Kepler}
\]

\[
+ \frac{3}{4}(\mu/r)(R_{\oplus}/r)^2 J_2 \sin^2 I \quad \text{Main equatorial}
\]

\[
- \frac{3}{4}(\mu/r)(R_{\oplus}/r)^2 J_2 \sin^2 I \cos 2\theta \quad \text{Cid intermediary}
\]

\[
\text{Full problem}
\]

- Radial Hamiltonians: \( (r, \theta, \nu, R, \Theta, N) \rightarrow (r, -, -, R, \Theta, N) \)
  - \( \mathcal{H}(r, R; \Theta, N) \) 1-DOF \( \Rightarrow \) integrable

- 3 different intermediaries of the main problem
• Good and bad radial intermediaries (classical definition)
  - Keplerian: \( \mathcal{K} = \frac{1}{2} \left( R^2 + \frac{\Theta^2}{r^2} \right) - \frac{\mu}{r} \), \( \frac{1}{2\pi} \int_0^{2\pi} (\mathcal{H} - \mathcal{K}) dM \neq 0 \)
  - Equatorial main problem \( \mathcal{E} = \mathcal{K} + \mathcal{P}_Q \)
    \[ \mathcal{E} = \mathcal{K} - \frac{\mu}{2r} \frac{R_\oplus^2}{r^2} J_2 \]
    \[ \frac{1}{2\pi} \int_0^{2\pi} (\mathcal{H} - \mathcal{E}) dM \neq 0 \]
  - Cid-Lahulla intermediary \( \mathcal{C} = \mathcal{K} + \mathcal{P}_Q + \mathcal{P}_R \),
    \[ \mathcal{C} = \mathcal{K} - \frac{\mu}{2r} \frac{R_\oplus^2}{r^2} J_2 \left( 1 - \frac{3}{2}\sin^2 I \right) \]
    \[ \frac{1}{2\pi} \int_0^{2\pi} (\mathcal{H} - \mathcal{C}) dM = 0 \]

• Cid-Lahulla: paradigm of common intermediaries
  - \( \langle \mathcal{C} \rangle \equiv \langle \mathcal{H} \rangle \) same secular rates as the main problem
  - actual orbit: short-period oscillations about Cid’s orbit
    * except for effects of the 2nd order of \( J_2 \)
  - analytical solution in elliptic integrals
Zonal intermediaries

- Add & subtract $A$ to the main problem $\mathcal{H}$
  - reorganize: $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ \quad ($s \equiv \sin I, \ c \equiv \cos I$)

$$\begin{align*}
\mathcal{H}_0 &= \frac{1}{2} \left( R^2 + \frac{\Theta^2}{r^2} \right) - \frac{\mu}{r} - \frac{\mu}{2p} \frac{R_\oplus^2}{r^2} J_2 \left( 1 - \frac{3}{2}s^2 + \frac{3}{2}s^2 \cos 2\theta \right) \\
\mathcal{H}_1 &= -\frac{\mu}{2p} \left( \frac{p}{r} - 1 \right) \frac{R_\oplus^2}{r^2} J_2 \left( 1 - \frac{3}{2}s^2 + \frac{3}{2}s^2 \cos 2\theta \right); \quad \frac{p}{r} - 1 = e \cos f
\end{align*}$$

- $\mathcal{H}_0$ integrable (elliptic integrals); $\langle \mathcal{H}_1 \rangle = 0$
  * (Aksnes 1965, Astrophisica Norvegica)
- low $e$: $e \sim \mathcal{O}(J_2) \Rightarrow \mathcal{H}_1 = \mathcal{O}(J_2^2)$

- Vinti 1959 JR-NBS, Aksenov et al. 1961 P&SS:
  - accurate up to (some) 2nd order effects of $J_2$
Natural intermediaries

- Integrable after a contact transform. (Deprit 1981 CeMDA)
  \[ (r, \theta, \nu, R, \Theta, N) \rightarrow (r', \theta', \nu', R', \Theta', N') \]
- Accurate up to \( O(J_2) \) secular and periodic effects

- Most of the common intermediaries can be naturalized
  - Cid-Lahulla, Aksnes, . . . : solution in elliptic integrals

- Deprit’s radial intermediary:
  \[
  H = \frac{1}{2} \left( R'^2 + \frac{\Theta'^2}{r'^2} \right) - \frac{\mu}{r'}, \quad \Theta' = \Theta' \sqrt{1 + J_2 \frac{R^2}{p'^2} \left( \frac{1}{2} - \frac{3}{2} c'^2 \right)}
  \]
  - Quasi-Keplerian system with variable angular momentum
  - Solution in trigonometric functions
  - Very simple periodic corrections
\[
\xi - \xi' = -(1/2)J_2 (R_\oplus/p)^2 \Delta \xi, \quad \xi \in (r, \theta, \nu, R, \Theta, N),
\]
\[
\Delta N = 0
\]
\[
\Delta r = (1/2)p \left( 1 - 3s^2 - s^2 \cos 2\theta \right)
\]
\[
\Delta \theta = \left( pR/\Theta \right) \left[ 1 - 6c^2 + (1 - 2c^2) \cos 2\theta \right]
+ (1/4) \left[ 3 - 5c^2 - 4(1 - 3c^2)(p/r) \right] \sin 2\theta
\]
\[
\Delta \nu = (1/2) c \left[ (pR/\Theta)(6 + 2 \cos 2\theta) + (1 - 4p/r) \sin 2\theta \right]
\]
\[
\Delta R = \left( p\Theta/r^2 \right) s^2 \sin 2\theta
\]
\[
\Delta \Theta = (1/2) \Theta s^2 \left[ (1 - 4p/r) \cos 2\theta - (pR/\Theta) \sin 2\theta \right]
\]
DRI: 2nd order improvements

• 2nd order transform. \((r, \theta, \nu, R, \Theta, N) \rightarrow (r', \theta', \nu', R', \Theta', N')\)
  – computed by “elimination of the parallax”

• New Hamiltonian term
  \[- H_{0,2} = \frac{1}{4} J_2^2 \frac{\Theta^2}{r^2} \left( \Phi_1 + \frac{J_3}{J_2^2} \Phi_2 + \frac{J_4}{J_2^2} \Phi_3 \right) \]
  – \(\Phi_m = \Phi_m(p, e, i, \omega)\)
  – \(p = p(\Theta), e \equiv e(r, -, R, \Theta), i \equiv i(\Theta, N), \omega \equiv \omega(r, \theta, R, \Theta)\)

• No longer integrable, but \(H_{0,2} = \Psi(r, -, R, \Theta) + O(eJ_2^2)\)

• \(e\) small: \(H_{0,2} \approx \Psi(r, -, R, \Theta)\)
  – again radial (integrable) and quasi-Keplerian!!
  – limited to low \(e\) . . . most common case in LEO
  – new periodic corrections more involved, yet manageable
LEO Performance: examples

• Spot-type satellite:
  – $a = 7081.139$ km, $e = 0.0158$, $i = 98^\circ$,
  – $\Omega = 164.02^\circ$, $\omega = M = 0$

• Test cases for one day:
  – Numerical integration of the $J_2$–$J_4$ problem
  – Numerical integration of the $J_2$ (main) problem
  – Quasi-Keplerian intermediary in mean elements $+$ . . .
    * full 2nd order inverse & direct transformation eqs.
    * simplified, 2nd order inverse $+$ 1st order direct eqs.
- $J_2 - J_4$ model vs.: $J_2$-numerical and intermediary (full)
- $J_2 - J_4$ vs. $J_2$ numeric & intermediary (simp.): 3 times faster
$J_3$ long-period effects

- Clearly noted since the beginning of the propagation
  - perigee dynamics, $e, i$ dynamics
  - intermediary only deals with $J_3$ short-period effects
- New canonical transformation contrary to truncation
  - Alfriend & Coffey’s *elimination of the perigee* (1984)
  - extremely simple formulas for the case of LEO
- Sequence:
  - short-period inverse corrections ($J_2$, $J_2^2$, $J_3$, $J_4$)
  - long-period inverse corrections ($J_3$)
  - quasi-Keplerian intermediary evaluation ($J_2$, $J_2^2$ and $J_4$)
  - long-period direct corrections ($J_3$)
  - short-period inverse corrections ($J_2$)
• **without** long-period corrections

![Graphs showing deviations in 
\( \Delta e \sin \omega \) and 
\( \Delta e \cos \omega \) without corrections.]

• **with** \( J_3 \) long-period corrections: observable improvements

![Graphs showing deviations in 
\( \Delta e \sin \omega \) and 
\( \Delta e \cos \omega \) with corrections.]

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<th>Days</th>
<th>0.0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
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<td>( \Delta e \sin \omega )</td>
<td>-6.0 \times 10^{-6}</td>
<td>-4.0 \times 10^{-6}</td>
<td>-2.0 \times 10^{-6}</td>
<td>0</td>
<td>2.0 \times 10^{-6}</td>
<td>4.0 \times 10^{-6}</td>
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<tr>
<td>( \Delta e \cos \omega )</td>
<td>0.00005</td>
<td>0.00003</td>
<td>0.00001</td>
<td>0</td>
<td>0.00003</td>
<td>0.00005</td>
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<tr>
<td>( \Delta \theta ) (arc seconds)</td>
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<td>-0.02</td>
<td>0</td>
<td>0.02</td>
<td>0.05</td>
<td>0.08</td>
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9-1
• 4 months **without** long-period corrections

• 4 months **with** \( J_3 \) long-period corrections
Conclusions

- Higher order geopotential: improves propagation of LEOs
  - penalizes Cowell integration in terms of computing time
- Increase in computational burden $\rightarrow$ power consumption
  - can be radically alleviated for the lower eccentricity orbits
  - intermediary solution, within a reasonable accuracy.
    * neglect terms $O(e^2J_2^2)$ of the perigee dynamics
- Our intermediary: higher order secular and periodic effects
  - compact form of straightforward evaluation (polar vars.)
  - useful for onboard orbit propagation: restricted power
- Future improvements and work
  - include atmospheric drag effects
  - try other intermediaries: Aksnes, Vinti, 2 fixed centers