A FAST AND EFFICIENT ALGORITHM FOR ONBOARD LEO INTERMEDIARY PROPAGATION

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ABSTRACT

A new intermediary solution to the earth satellite problem is discussed. The intermediary takes into account the first three zonal harmonics of the Geopotential. It is shown that this analytical solution can advantageously replace the Cowell integration of the J_2 problem which is customarily used onboard for short-term prediction of low earth orbits.

Index Terms— Low earth orbit, Geopotential, natural intermediary, Lie transforms, elimination of the parallax

1. INTRODUCTION

Usual onboard orbit propagators are provided as navigation maintenance aids for earth satellites. These programs should be able to forecast satellite ephemeris within a reasonable accuracy for short time periods, which may range from minutes, as in the case of momentary lack of GPS signal, to several satellite orbits. In these brief intervals the accumulation of second order effects of the Geopotential is barely apparent, and, therefore, the propagation model can very simple. Hence, common onboard orbit propagators are based in the fixed-step numerical integration of the J_2 model, that is, the earth's zonal Geopotential truncated to the zonal harmonic of the second degree.

On the other hand, the use of analytical, intermediary solutions of the J_2 problem has been recently proposed as an efficient alternative to the numerical integration. The accuracy of common intermediary orbits of the J_2 problem is limited to first order effects, thus providing less precise solutions than the numerical integration. However, because of the inherent uncertainty of the initial conditions to be propagated onboard, it can be shown that both alternatives, the numerical integration and the intermediary approach, enjoy the same statistics [1]. Other benefits of using analytical solutions is that they may improve both memory allocation and computation time, a fact that can be crucial to Cubesats ore other small satellite missions, in which the computational abilities may be restricted.

Note, however, that neglecting the long-period effects associated to the odd zonal harmonics introduces small errors in the propagation, which are clearly observable even in the short-term. These errors can exceed 1 km in the along-track direction at the end of one day. Hence, taking into account the disturbing effects of some higher order harmonics may notably improve the propagation model.

Here, we propose a new intermediary solution that takes into account the first three zonal harmonics of the Geopotential (J_2, J_3 and J_4). Since the solution is analytical, its evaluation is very fast and is not constrained to a step-by-step evaluation. In spite of the forces model of the new intermediary is much heavier than the simple J_2 model, its evaluation can be fastened using some simplifications that alleviate the computational burden, in this way making the new intermediary definitely competitive when compared to the numerical integration of the J_2 problem.

2. LEO INTERMEDIARY SOLUTION

We study the motion of the satellite, considered as a mass point, under the disturbing effects of the earth's gravitational potential. In particular, our model takes into account the effects of the zonal harmonics J_2, J_3, and J_4. This model is of limited accuracy, as shown in Fig. 1, where the clear increase of the errors in the tangent direction is mostly related with the drag perturbation. Nevertheless, it would improve predictions in all those cases in which the simpler J_2 model is currently been used, as in onboard orbit propagation.

The problem is written using Hamiltonian formulation

$$\mathcal{H} = \mathcal{H}_{\mathrm{K}} + \frac{\mu}{r} \sum_{m \geq 2} \left(\frac{\alpha}{r}\right)^m C_{m,0}\, P_m(\sin\varphi), \tag{1}$$

where $\mathcal{H}_{\mathrm{K}}$ represents the Keplerian attraction, and the disturbing function encompasses the non-centralities of the gravitational potential, where μ is the earth's gravitational parameter, r is the distance from the earth's center of mass, φ is

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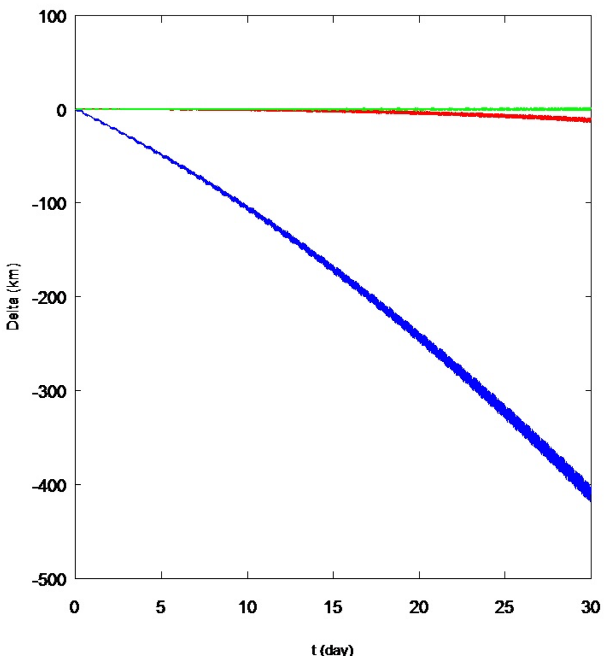

Fig. 1. Radial (red lines), normal (green lines), and tangent errors (blue lines) between the J_2–J_4 model and a realistic model for a sample propagation of a Spot-type satellite.

geographic latitude, the scaling factor α is the earth's equatorial radius, P_m are Legendre polynomials of degree m, and $C_{m,0} = -J_m$ are corresponding zonal harmonic coefficients.

The differential equations of the flow corresponding to the zonal Hamiltonian (1) are obtained from Hamilton equations. Thus, the variation of each coordinate is given by the partial derivative of the Hamiltonian with respect to its conjugate momentum, whereas the variation of each momentum is given by *minus* the partial derivative of the Hamiltonian with respect to its conjugate coordinate. This flow is not integrable, in general, but, in view of the smallness of the harmonic coefficients, approximate solutions can be found using perturbation theory.

Thus, Eq. (1) is simplified by a canonical transformation $(\boldsymbol{x}, \boldsymbol{X}) \longrightarrow (\boldsymbol{x}', \boldsymbol{X}')$, where $\boldsymbol{x}$ are coordinates and $\boldsymbol{X}$ their conjugate momenta, from osculating to new (prime) variables. In particular, after carrying out the elimination of the parallax simplification [2, 3, 4], up to the second order of $C_{2,0}$ we get the Hamiltonian in prime variables

$$\begin{aligned} \mathcal{K} = & -\frac{\mu}{2a} + \frac{\mu}{2p}\frac{\alpha^2}{r^2}C_{2,0}\left(1 - \frac{3}{2}s^2\right) + \frac{\mu}{2p}\frac{p^2}{r^2} \\ & \times\Bigg\{ -\frac{3}{4}\frac{\alpha^3}{p^3}es\left(1-5c^2\right)C_{3,0}\sin\omega + \frac{1}{4}\frac{\alpha^4}{p^4}C_{2,0}^2 \\ & \times\Bigg[\left(\frac{1}{4} - \frac{21}{4}c^4\right) - \frac{3}{2}e^2\left(c^2 - \frac{5}{8}s^4\right) \\ & -\frac{3}{8}\left(1-15c^2\right)e^2s^2\cos 2\omega\Bigg] - \frac{3}{4}\frac{\alpha^4}{p^4}C_{4,0} \\ & \times\Bigg[\left(1 - 5s^2 + \frac{35}{8}s^4\right)\left(1 + \frac{3}{2}e^2\right) \\ & -\frac{5}{8}\left(1-7c^2\right)e^2s^2\cos 2\omega\Bigg]\Bigg\}, \end{aligned} \tag{2}$$

where, because of the Hamiltonian dynamics, all the elements, viz. the semi-major axis a, the conic parameter p, the eccentricity e, the radius r, the argument of the perigee ω, and the inclination, given by $s \equiv \sin I$, and $c \equiv \cos I$, are not variables but functions of some canonical set of variables.

Specifically, we use Delaunay variables (ℓ, g, h, L, G, H), standing for the mean anomaly, the argument of the perigee, the right ascension of the ascending node, the Delaunay action, the modulus of the angular momentum vector, and the projection of the angular momentum vector on the earth's rotation axis, respectively. Then,

$$a = \frac{L^2}{\mu}, \tag{3}$$
$$\omega = g, \tag{4}$$
$$s = \sqrt{1-c^2}, \tag{5}$$
$$c = \frac{H}{G}, \tag{6}$$
$$e = \sqrt{1-\eta^2}, \tag{7}$$
$$\eta = \frac{G}{L}, \tag{8}$$
$$p = a\eta^2 = \frac{G^2}{\mu}, \tag{9}$$
$$r = \frac{p}{1+e\cos f}, \tag{10}$$

where

$$f \equiv f(\ell, -, -, L, G, -), \tag{11}$$

is the true anomaly, which is an implicit function of ℓ that involves the solution of the Kepler equation.

For the lower eccentricity orbits $e\,C_{2,0}^2 = \mathcal{O}(C_{2,0}^3)$, and corresponding terms may be neglected from Eq. (2). In particular, this simplification prevents the appearance of the argument of the perigee in the Hamiltonian, which, by this reason, turns out to be integrable.

Indeed, using polar-nodal variables $(r, \theta, \nu, R, \Theta, N)$, where r has been defined in Eq. (10) and must be expressed now in prime variables $r \equiv r(\ell', -, -, L', G', -)$,

$$\theta = f(\ell' -, -, L', G', -) + g', \tag{12}$$

is the argument of the latitude, $\nu = h'$,

$$R = \sqrt{\frac{\mu}{a}}\frac{e}{\eta}\sin f, \tag{13}$$

is the radial velocity, also expressed in prime variables $R \equiv R(\ell', -, -, L', G', -)$, $\Theta = G'$, and $N = H'$, the simplified Hamiltonian after neglecting the second order terms which have the eccentricity as a factor, is written

$$\mathcal{K} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} + \frac{1}{2}\frac{\Theta^2}{r^2}\epsilon\left(2 - 3s^2\right) \tag{14}$$
$$+\frac{1}{2}\frac{\Theta^2}{r^2}\epsilon^2\left[\left(\frac{1}{4} - \frac{21}{4}c^4\right) - 3\frac{C_{4,0}}{C_{2,0}^2}\left(1 - 5s^2 + \frac{35}{8}s^4\right)\right]$$

where we abbreviated

$$\epsilon = \frac{1}{2}\frac{\alpha^2}{p^2}C_{2,0}, \tag{15}$$

and now, in the polar-nodal variables, $c = N/\Theta$ and $p = \Theta^2/\mu$. Note that, because the argument of the periapsis has been removed from the Hamiltonian, θ is cyclic in Eq. (14) and, therefore, Θ is constant and so it is $\epsilon \equiv \epsilon(\Theta)$.

The transformation equations leading to Eq. (2), and consequently to the simplified Hamiltonian (14), are given in A, where terms of the order of $e^2 C_{2,0}^2$ have been neglected in agreement with the aimed accuracy of the intermediary solution.

Equation (14) may be reorganized as

$$\mathcal{K} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\Phi^2\right) - \frac{\mu}{r}, \tag{16}$$

where $\Phi \equiv \Phi(\Theta, N)$ is given by

$$\Phi^2 = 1 + \epsilon\left(2 - 3s^2\right) \tag{17}$$
$$+\epsilon^2\left[\left(\frac{1}{4} - \frac{21}{4}c^4\right) - 3\frac{C_{4,0}}{C_{2,0}^2}\left(1 - 5s^2 + \frac{35}{8}s^4\right)\right]$$

which is also constant. The new Hamiltonian in Eq. (16) represents a *quasi Keplerian* system with varied "angular momentum" $\tilde{\Theta} = \Theta\Phi$, and it can be integrated by the usual Hamiltonian reduction in Delaunay variables. The sequence is as follows (see [1] for further details).

1. Compute the constants $c = N/\Theta$, $s = (1-c^2)^{1/2}$, and:
   (a) evaluate ϵ from Eq. (15), with $p = \Theta^2/\mu$;
   (b) evaluate Φ in Eq. (17), and $\mathcal{K} = E$ in Eq. (16);
   (c) make $\tilde{a} = -\mu/(2E)$, $\tilde{p} = (\Theta\Phi)^2/\mu$, and $\tilde{e} = (1 - \tilde{p}/\tilde{a})^{1/2}$.
2. Starting from the polar-nodal variables and $t = t_0$:
   (a) solve the auxiliary variable ϕ from
   $$\tilde{r} = \frac{\tilde{p}}{1 + \tilde{e}\cos\phi}, \qquad \tilde{R} = \sqrt{\frac{\mu}{\tilde{p}}}\,\tilde{e}\sin\phi; \tag{18}$$
   (b) compute the auxiliary variable u from
   $$\tan\frac{u}{2} = \sqrt{\frac{1-\tilde{e}}{1+\tilde{e}}}\tan\frac{\phi}{2}; \tag{19}$$
   (c) finally, evaluate
   $$\lambda = u - \tilde{e}\sin u, \tag{20}$$
   $$\gamma = \theta - \frac{\phi}{\Phi}\Big\{1 + \epsilon\,(1 - 6c^2) - \frac{3}{8}\epsilon^2 \times \Big[\tag{21}$$
   $$2 - 70c^4 - (3 - 35c^2)(3 - 5c^2)\tilde{C}_{4,0}\Big]\Big\},$$
   $$h = \nu - 3\epsilon\frac{\phi}{\Phi}c \tag{22}$$
   $$\times\left\{1 - \left[\frac{7}{2}c^2 - \frac{5}{4}(3 - 7c^2)\tilde{C}_{4,0}\right]\epsilon\right\},$$
   where $\tilde{C}_{4,0} = C_{4,0}/C_{2,0}^2$.
3. Then, for a given time t:
   (a) evaluate $\lambda = \lambda(t_0) + \sqrt{\mu/\tilde{a}^3}t$;
   (b) solve Eq. (20) for u and compute ϕ from Eq. (19)
   (c) evaluate r and R from Eq. (18), and solve θ from Eq. (21), and ν from Eq. (22).

3. NUMERICAL EXPERIMENTS

In order to check the usefulness of the intermediary solution, we compare it with the numerical integration of the original problem in Cartesian coordinates —the flow derived from Eq. (1) with $m = 4$— for a variety of test cases.

First of all, we illustrate the effects of neglecting second order terms of the Earth gravitational potential for a Eyesat-type satellite, which is a Cubesat. We use the initial conditions corresponding to the following orbital elements

$$\begin{aligned} a &= 7078.0\,\mathrm{km}, \\ e &= 0.00001, \\ i &= 98.18°, \\ \Omega &= 0, \\ \omega &= 0, \\ M &= 0, \end{aligned}$$

and propagate them for 4 months (or about 1750 orbital periods). Errors between the full zonal model propagation and the J_2 truncation are shown in Fig. 2 (blue lines), where corresponding errors between the full zonal model propagation and the intermediary propagation (red lines) have been superimposed. Note that because of the low eccentricity of the Eyesat orbit, instead of providing errors for the mean anomaly, the argument of the perigee and the eccentricity, in order to avoid additional errors introduced by the inaccurate determination of the argument of perigee, we provide errors for the usual alternative elements [5]

$$F = M + \omega, \qquad C = e\cos\omega, \qquad S = e\sin\omega. \tag{23}$$

The effects of neglecting the long-period terms of the J_3 contribution are clearly apparent in Fig. 2. In particular, a long-period modulation of the errors of the intermediary propagation with the same period as the argument of the perigee (of about 16 weeks and a half) are clearly noted. In the case of the elements C and S, the errors are almost the same in the J_2 propagation and in the intermediary evaluation, although for the later the amplitude of the short-period terms is almost negligible, in consequence with the short-period corrections used by the intermediary. On the contrary, the contribution of J_4 makes that errors introduced by the J_2 truncation are unacceptable, and are particularly evident in the evolution of the errors of Ω and the element F. On the other hand, when using the intermediary solution the errors for these two elements are almost negligible, falling below 4 arc seconds in any case.

Similar tests have been performed for a variety of orbits, always finding analogous results. In particular, different simulations have been carried out for the parameters of a typical common LEO, with orbital elements

$$\begin{aligned} a &= 6831.5723\,\mathrm{km}, \\ e &= 0.001357, \\ i &= 51.6^\circ, \\ \Omega &= 224.8^\circ, \\ \omega &= 280.1^\circ, \\ M &= 66.5^\circ. \end{aligned}$$

Sample errors for one day propagation are presented in Fig. 3; the errors in the intermediary and Runge-Kutta propagations overlap each other for the elements $e\cos\omega$ and $e\sin\omega$, and are not presented .

4. CONCLUSIONS

A compact analytical solution of the zonal problem has been obtained, which neglects secular and periodic effects of the third order of J_2 as well as long-period effects of the order $\mathcal{O}(e\,J_2^2)$. This intermediary solution clearly improves the short-term propagation of LEO orbits when compared to the Cowell step by step numerical integration of the J_2 problem, both in accuracy and evaluation speed. Hence it may be adequate for onboard orbit propagation in such satellite missions in which reduced power consumption is a constraint.

On the other hand, the zonal intermediary provided here misses long-period effects related to the perigee dynamics. However, an additional canonical transformation that removes, contrary to neglects, the perigee dynamics may be implemented, in this way allowing to cope with these long-period effects. An improved zonal intermediary that properly deals with the long-period terms associated to the J_3 dynamics is under development and corresponding results will be reported elsewhere.

A. THE ELIMINATION OF THE PARALLAX TRANSFORMATION

The elimination of the parallax is a canonical transformation depending on a small parameter ε

$$\mathcal{P} : (\boldsymbol{x}, \boldsymbol{X}) \longrightarrow (\boldsymbol{x}', \boldsymbol{X}', \varepsilon)$$

from original to "prime" variables, where $\boldsymbol{x}$ are coordinates and $\boldsymbol{X}$ their conjugate momenta, such that, up to some truncation order $\mathcal{O}(\varepsilon^m)$, it removes parallactic terms from the zonal Hamiltonian, in which case $\varepsilon = J_2$ [2, 3, 4].

Calling ξ to any of the canonical variables and using $\epsilon = \frac{1}{2}(\alpha/p)^2 C_{2,0}$, as given in Eq. (15), the direct transformation is written

$$\xi = \xi' + \epsilon\,\Delta_1\xi' + \frac{1}{2}\epsilon^2\,\delta_1\xi' + \mathcal{O}(\epsilon^3),$$

and the inverse transformation

$$\xi' = \xi + \epsilon\,\Delta_2\xi + \frac{1}{2}\epsilon^2\,\delta_2\xi + \mathcal{O}(\epsilon^3).$$

The corrections are naturally expressed in polar nodal variables and are given below, where the eccentricity functions

$$\kappa = \frac{p}{r} - 1, \qquad \sigma = \frac{p\,R}{\Theta}, \tag{24}$$

with $p = \Theta^2/\mu$, are used for convenience. Besides, the notation $\tilde{C}_{n,0} = C_{n,0}/C_{2,0}^2$ is used in the second order corrections.

Note that the first order corrections were first provided by [2], and are given here for the sake of completeness —yet in the arrangement proposed in [1] whose numerical evaluation is much more efficient. Besides second order terms corresponding to the J_2 perturbation have been previously given in [6]. Note also that terms of the order of the square of the eccentricity have been neglected from the second order corrections, which shortens the correction series, in this way fastening their evaluation and improving the intermediary's performance, and is consistent with the assumptions of neglecting from the intermediary the long-period effects related to the perigee dynamics.

First order corrections

The first-order corrections are formally the same both for the direct and inverse transformations but with different signs, namely $\Delta_1 = \Delta$ and $\Delta_2 = -\Delta$ where:

$$\Delta r = p\left(1 - \frac{3}{2}s^2 - \frac{1}{2}s^2\cos 2\theta\right), \tag{25}$$

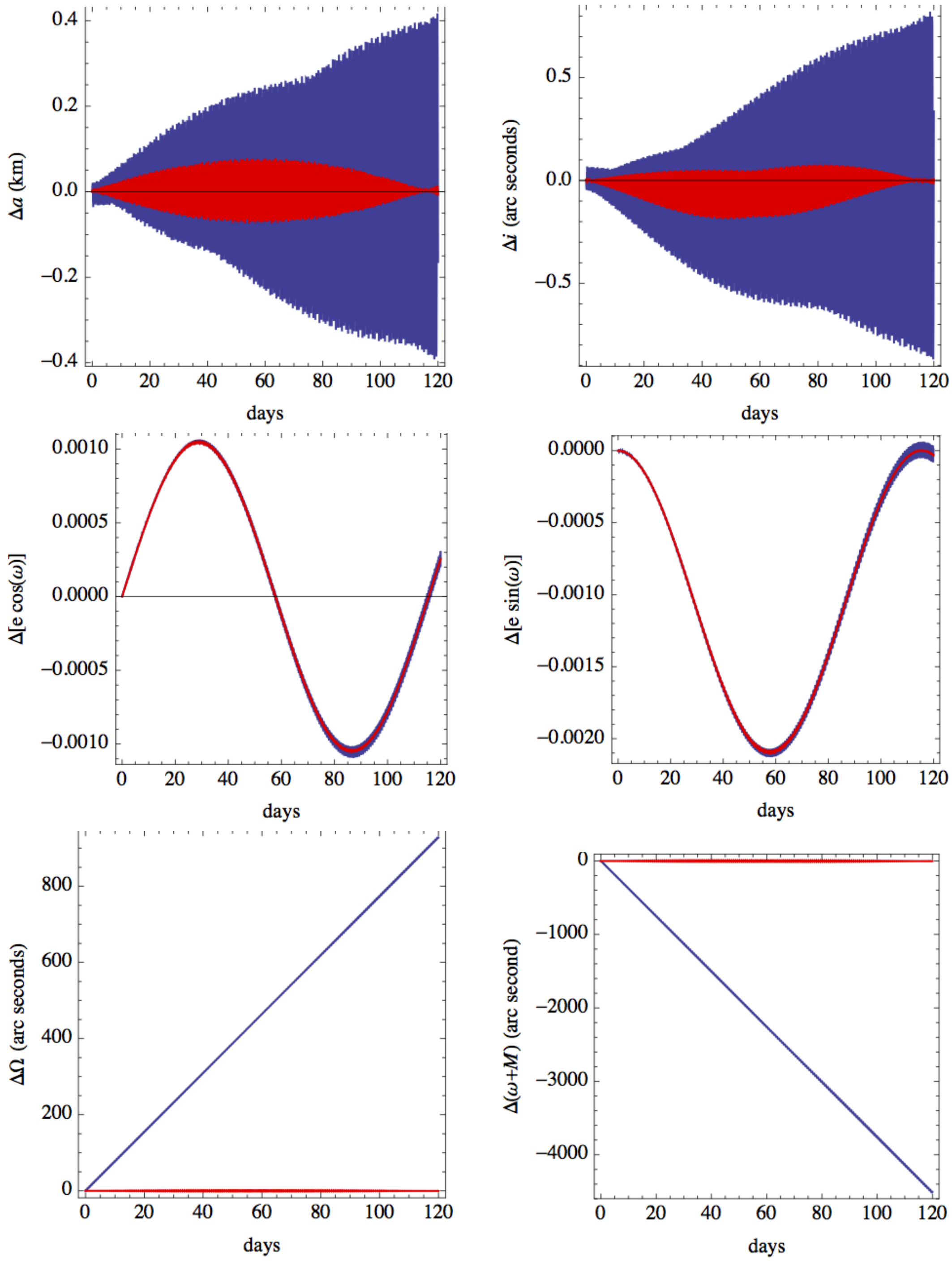

Fig. 2. Errors between the main problem and the full zonal model (blue lines) and between the intermediary and the full zonal model (red lines) for a EYESAT-type satellite.

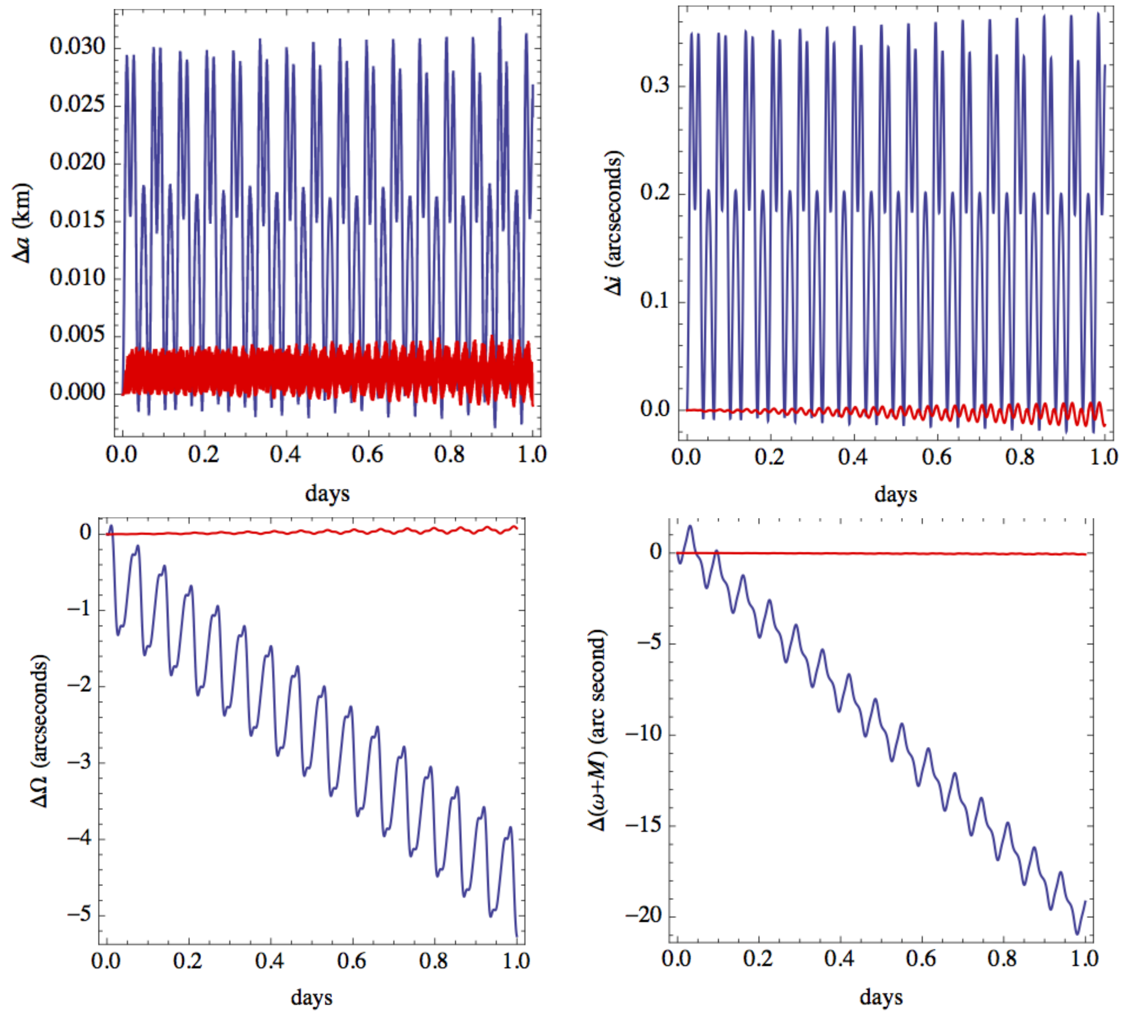

Fig. 3. Errors between the main problem and the full zonal model (blue lines) and between the intermediary and the full zonal model (red lines) for one day propagation of a typical LEO.

$$\begin{aligned}\Delta\theta &= \left[1-6c^2+(1-2c^2)\cos 2\theta\right]\sigma \\ &\quad -\left[\frac{1}{4}-\frac{7}{4}c^2+(1-3c^2)\,\kappa\right]\sin 2\theta,\end{aligned} \tag{26}$$

$$\Delta\nu = c\left[(3+\cos 2\theta)\,\sigma-\left(\frac{3}{2}+2\kappa\right)\sin 2\theta\right], \tag{27}$$

$$\Delta R = \frac{\Theta}{p}\,(1+\kappa)^2 s^2\sin 2\theta, \tag{28}$$

$$\Delta\Theta = -\Theta\, s^2\left[\left(\frac{3}{2}+2\kappa\right)\cos 2\theta+\sigma\sin 2\theta\right], \tag{29}$$

$$\Delta N = 0, \tag{30}$$

where the right member of each of Eqs. (25)–(30) as well as p in Eq. (15) must be expressed in prime variables when computing $\Delta_1\xi' = \Delta\xi'$, or in original ones when computing $\Delta_2\xi = -\Delta\xi$.

Second order direct corrections

The right sides are assumed to be expressed in prime variables.

$$\begin{aligned}\frac{\delta_1 r}{p'} &= \frac{1}{4}(5-14c^2-23c^4)-\frac{1}{16}(9-26c^2+41c^4)\kappa \\ &\quad +\frac{1}{16}\sigma(3+51c^2)s^2\sin 2\theta+\left[1-14c^2-\frac{1}{16}\kappa\right. \\ &\quad \left.\times(23-153c^2)\right]s^2\cos 2\theta-\frac{1}{16}(4-\kappa)s^4\cos 4\theta \\ &\quad +\frac{9}{32}\sigma s^4\sin 4\theta+\frac{p}{\alpha}\tilde{C}_{3,0}\left[\frac{4}{3}\sigma s^3\cos 3\theta+(4-5s^2)\right. \\ &\quad \times\left(\frac{3+4\kappa}{2}s\sin\theta-4\sigma s\cos\theta\right)-\frac{1}{4}(5+8\kappa)s^3 \\ &\quad \left.\times\sin 3\theta\right]+\tilde{C}_{4,0}\left[(1-7c^2)s^2\left(\frac{45}{16}\sigma\sin 2\theta-\frac{5}{16}\right.\right.\end{aligned}$$

$$\times(8-3\kappa)\cos 2\theta\Big) - \frac{9}{16}(2+\kappa)(3-30c^2 +35c^4) + \frac{7}{16}(2+5\kappa)s^4\cos 4\theta + \frac{35}{32}\sigma s^4 \sin 4\theta\Big]$$

$$\delta_1\theta = \frac{\sigma}{8}\Big[-65+314c^2+327c^4-(1609c^4-1240c^2 +79)\cos 2\theta + (1-26c^2+c^4)\cos 4\theta\Big] + \frac{1}{8}\Big[2(1-54c^2+85c^4)+(69-1232c^2+1419c^4)\kappa\Big] \times\sin 2\theta - \frac{1}{16}\Big[2(2-37c^2+17c^4)+(7-158c^2 +55c^4)\kappa\Big]\sin 4\theta + \frac{p}{\alpha}\tilde{C}_{3,0}\Big\{\frac{1}{4}\sigma\Big[\frac{1}{s}(2-26c^2) \times(4-5c^2)\sin\theta + (4-19c^2)s\sin 3\theta\Big] + \frac{1}{12}s \times\big[10(1-7c^2)+(23-158c^2)\kappa\big]\cos 3\theta - \frac{1}{2s} \times\big[6(4-35c^2+35c^4)+(1-39c^2+50c^4)\kappa\big] \times\cos\theta\Big\} + \tilde{C}_{4,0}\Big\{\sigma\Big[\frac{135}{8}(1-14c^2+21c^4) +\frac{5}{8}(27-280c^2+301c^4)\cos 2\theta - \frac{7}{8}(1-5c^2)s^2 \times\cos 4\theta\Big] + \frac{7}{32}\big[3-23c^2+2(5-37c^2)\kappa\big]s^2 \times\sin 4\theta + \frac{5}{8}\big[2(1+8c^2-21c^4)-(25-328c^2 +399c^4)\kappa\big]\sin 2\theta\Big\}$$

$$\delta_1\nu = c\Big\{\Big[4(23c^2-9)\cos 2\theta - \frac{27}{2}(1+c^2)+(1+c^2) \times\frac{3}{2}\cos 4\theta\Big]\sigma + \Big[13-21c^2+4(11-19c^2)\kappa\Big] \times\sin 2\theta - \frac{3}{4}(4-c^2+8\kappa)\sin 4\theta + \frac{p}{\alpha}\tilde{C}_{3,0}\times\Big[\Big(\frac{6}{s}-\frac{45}{2}s\Big)\Big[(2+\kappa)\cos\theta - \sigma\sin\theta\Big] + (4+9\kappa) \times\frac{5}{4}s\cos 3\theta + \frac{15}{4}\sigma s\sin 3\theta\Big] + \tilde{C}_{4,0}\Big[\Big(45(3-7c^2) +20(4-7c^2)\cos 2\theta - 7s^2\cos 4\theta\Big)\frac{\sigma}{2} - (1+4\kappa) \times 5(4-7c^2)\sin 2\theta + \frac{7}{8}(5+16\kappa)s^2\sin 4\theta\Big]\Big\}$$

$$\delta_1 R = \frac{\Theta}{p}\Big\{\Big[39-134c^2+71c^4+(41-231c^2)s^2\cos 2\theta +17s^4\cos 4\theta\Big]\frac{\sigma}{16} + \frac{1}{16}\big[256c^2+(37+437c^2)\kappa\big] \times s^2\sin 2\theta + \frac{1}{32}(32+65\kappa)s^4\sin 4\theta + \frac{p}{\alpha}\tilde{C}_{3,0} \times\Big[\frac{1}{2}(1-5c^2)(3+10\kappa)s\cos\theta - \frac{s^3}{12}(45+146\kappa) \times\cos 3\theta - 2(1-5c^2)\sigma s\sin\theta - 2\sigma s^3\sin 3\theta\Big] +\tilde{C}_{4,0}\Big[\frac{1}{16}(80+235\kappa)(1-7c^2)s^2\sin 2\theta -\frac{7}{32}(16+67\kappa)s^4\sin 4\theta + \Big(-9(3-30c^2+35c^4) +15(1-7c^2)s^2\cos 2\theta + 35s^4\cos 4\theta\Big)\frac{\sigma}{16}\Big]\Big\}$$

$$\delta_1\Theta = \Theta\Big\{-\frac{s^2}{4}\Big[7-25c^2+24(1-3c^2)\kappa+2(1+15c^2 +64c^2\kappa)\cos 2\theta\Big] - \frac{3}{4}s^4\cos 4\theta + 8(1-8c^2)\sigma s^2 \times\sin 2\theta + \frac{3}{2}\sigma s^4\sin 4\theta + \frac{p}{\alpha}\tilde{C}_{3,0}\Big[\frac{3}{2}\sigma(1-5c^2)s \times\cos\theta + \frac{15}{4}\sigma s^3\cos 3\theta + \frac{3}{2}(2+\kappa)(1-5c^2)s \times\sin\theta - \frac{5}{4}(4+9\kappa)s^3\sin 3\theta\Big] + \tilde{C}_{4,0}\Big[\frac{5}{2}(1+4\kappa) \times(-1+7c^2)s^2\cos 2\theta + \frac{7}{8}(5+16\kappa)s^4\cos 4\theta +5(-1+7c^2)\sigma s^2\sin 2\theta + \frac{7}{2}\sigma s^4\sin 4\theta\Big]\Big\}$$

Second order inverse corrections

Now, the right sides must remain in the original variables

$$\frac{\delta_2 r}{p} = -3+10c^2+c^4-\frac{1}{16}(39-134c^2+71c^4)\kappa -\frac{3}{16}\sigma(1+17c^2)s^2\sin 2\theta - \Big[4-32c^2+\frac{1}{16}\kappa \times(41-231c^2)\Big]s^2\cos 2\theta - \Big(1+\frac{17}{16}\kappa\Big)s^4\cos 4\theta -\frac{9}{32}\sigma s^4\sin 4\theta + \frac{p}{\alpha}\tilde{C}_{3,0}\Big[\frac{s}{2}\Big((3+4\kappa)\sin\theta - 8\sigma \times\cos\theta\Big)(1-5c^2) + \frac{1}{4}(5+8\kappa)s^3\sin 3\theta - \frac{4}{3}\sigma s^3 \times\cos 3\theta\Big] + \tilde{C}_{4,0}\Big[\frac{9}{16}(3-30c^2+35c^4)(2+\kappa)$$

$$+\frac{5}{16}(8-3\kappa)(1-7c^2)s^2\cos 2\theta - \frac{7}{16}(2+5\kappa)s^4 \times\cos 4\theta - \frac{45}{16}\sigma(1-7c^2)s^2\sin 2\theta - \frac{35}{32}\sigma s^4\sin 4\theta\bigg]$$

$$\delta_2\theta = \frac{\sigma}{8}(109-442c^2-243c^4)+\frac{\sigma}{8}(1993c^4-1928c^2 +143)\cos 2\theta + \frac{1}{8}\sigma(19-86c^2+43c^4)\cos 4\theta + \bigg[\frac{1}{4}(3+22c^2-73c^4) - \frac{1}{8}(1371c^4-1104c^2 +53)\kappa\bigg]\sin 2\theta + \frac{1}{16}\Big[2(19-17c^2+16c^4)+\kappa \times(23+98c^2-25c^4)\Big]\sin 4\theta + \frac{p}{\alpha}\tilde{C}_{3,0}\bigg\{-\frac{s}{12} \times\left[(23-158c^2)\kappa+10(1-7c^2)\right]\cos 3\theta -\frac{\sigma}{4}(4-19c^2)s\sin 3\theta - \frac{\sigma}{2s}(1-13c^2)(4-5c^2) \times\sin\theta + \frac{1}{2s}\Big[6(4-35c^2+35c^4)+(1-39c^2 +50c^4)\kappa\Big]\cos\theta\bigg\} + \tilde{C}_{4,0}\bigg\{-\frac{135}{8}\sigma(1-14c^2 +21c^4) - \frac{5}{8}\sigma(27-280c^2+301c^4)\cos 2\theta +\frac{7}{8}\sigma(1-5c^2)s^2\cos 4\theta - \frac{5}{4}\Big[1+8c^2-21c^4 -\frac{1}{2}(25-328c^2+399c^4)\kappa\Big]\sin 2\theta -\frac{7}{32}s^2\left[3-23c^2+2(5-37c^2)\kappa\right]\sin 4\theta\bigg\}$$

$$\delta_2\nu = c\bigg\{\frac{27}{2}\sigma(1+c^2)+2\sigma(33-46c^2)\cos 2\theta + \frac{3}{2}\sigma \times(3-c^2)\cos 4\theta - \Big[3(3-7c^2)+4(9-19c^2) \times\kappa\Big]\sin 2\theta - \frac{3}{4}(2+c^2+8\kappa)\sin 4\theta + \frac{p}{\alpha}\tilde{C}_{3,0} \times\bigg[\left(\frac{6}{s}-\frac{45s}{2}\right)\Big(\sigma\sin\theta-(2+\kappa)\cos\theta\Big) -5\left(1+\frac{9}{4}\kappa\right)s\cos 3\theta - \frac{15}{4}\sigma s\sin 3\theta\bigg] +\tilde{C}_{4,0}\bigg[-\frac{45}{2}\sigma(3-7c^2)-10\sigma(4-7c^2) \times\cos 2\theta + \frac{7}{2}\sigma s^2\cos 4\theta + 5(1+4\kappa)(4-7c^2) \times\sin 2\theta - \left(\frac{35}{8}+14\kappa\right)s^2\sin 4\theta\bigg]\bigg\}$$

$$\delta_2 R = \frac{\Theta}{p}\bigg\{\frac{\sigma}{16}(9-26c^2+41c^4)+\frac{\sigma}{16}(23-153c^2) \times s^2\cos 2\theta - \frac{\sigma}{16}s^4\cos 4\theta + s^2\Big[2(1-11c^2) +\frac{1}{16}(59-725c^2)\kappa\Big]\sin 2\theta + \left(1+\frac{95}{32}\kappa\right) \times s^4\sin 4\theta + \frac{p}{\alpha}\tilde{C}_{3,0}\bigg[\frac{1}{2}(1-5c^2)s\Big(4\sigma\sin\theta -(3+10\kappa)\cos\theta\Big) + 2\sigma s^3\sin 3\theta + \bigg(\frac{15}{4} +\frac{73}{6}\kappa\bigg)s^3\cos 3\theta\bigg] + \tilde{C}_{4,0}\bigg[\frac{9\sigma}{16}(3-30c^2 +35c^4) - \frac{15}{16}\sigma(1-7c^2)s^2\cos 2\theta - \frac{35}{16}\sigma s^4 \times\cos 4\theta - \frac{5}{16}(1-7c^2)(16+47\kappa)s^2\sin 2\theta +\frac{7}{32}(16+67\kappa)s^4\sin 4\theta\bigg]\bigg\}$$

$$\delta_2\Theta = \Theta\bigg\{-\left[\frac{1}{4}(7-25c^2)+6(1-3c^2)\kappa\right]s^2 -\left[\frac{3}{2}(1-9c^2)+(4-44c^2)\kappa\right]s^2\cos 2\theta +\frac{3}{4}s^4\cos 4\theta - \sigma(2-28c^2)s^2\sin 2\theta - \frac{3}{2}\sigma \times s^4\sin 4\theta + \frac{p}{\alpha}\tilde{C}_{3,0}\bigg[-\frac{3}{2}(1-5c^2)s\Big(\sigma \times\cos\theta + (2+\kappa)\sin\theta\Big) + \frac{5}{4}(4+9\kappa)s^3 \times\sin 3\theta - \frac{15}{4}\sigma s^3\cos 3\theta\bigg] + \tilde{C}_{4,0}\bigg[\frac{5}{2} \times(1-7c^2)s^2\Big(2\sigma\sin 2\theta + (1+4\kappa)\cos 2\theta\Big) -\frac{7}{8}(5+16\kappa)s^4\cos 4\theta - \frac{7}{2}\sigma s^4\sin 4\theta\bigg]\bigg\}$$

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