# A SEMI-ANALYTICAL ORBIT PROPAGATOR PROGRAM FOR HIGHLY ELLIPTICAL ORBITS*

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## ABSTRACT

The algorithms used in the construction of a semi-analytical propagator for the long-term propagation of Highly Elliptical Orbits (HEO) are described. The software propagates mean elements and include the main gravitational and non-gravitational effects that may affect common HEO orbits, as, for instance, geostationary transfer orbits or Molniya orbits.

***Index Terms***— HEO, Geopotential, third-body perturbation, tesseral resonances, SRP, atmospheric drag, mean elements, semi-analytic propagation

## 1. INTRODUCTION

A semi-analytical orbit propagator to study the long-term evolution of spacecraft in Highly Elliptical Orbits is presented. The perturbation model taken into account includes the gravitational effects produced by the first nine zonal harmonics and the main tesseral harmonics affecting to the 2:1 resonance, which has an impact on Molniya orbit-types, of Earth's gravitational potential, the mass-point approximation for third body perturbations, which only include the Legendre polynomial of second order for the sun and the polynomials from second order to sixth order for the moon, solar radiation pressure and atmospheric drag. Hamiltonian formalism is used to model the forces of gravitational nature so as to avoid time-dependence issues the problem is formulated in the extended phase space. The solar radiation pressure and the atmospheric drag are added as generalized forces. The semi-analytical theory is developed using perturbation techniques based on Lie transforms. Deprit's perturbation algorithm is applied up to the second order of the second zonal harmonics, $J_2$, including Kozay-type terms in the mean elements Hamiltonian to get "centered" elements. The transformation is developed in closed-form of the eccentricity except for tesseral resonances and the coupling between $J_2$ and moon's disturbing effects are neglected. The paper outlines the semi-analytical theory.

## 2. DYNAMICS OF A SPACECRAFT IN HEO

Satellites in earth's orbits are affected by a variety of perturbations of a diverse nature. A full account of this can be found in textbooks on orbital mechanics like [1]. All known perturbations must be taken into account in orbit determination problems. But for orbit prediction the accuracy requirements are notably relaxed, and hence some of the disturbing effects may be considered of higher order in the perturbation model.

Furthermore, for the purpose of long-term predictions it is customary to ignore short-period effects, which occur on time-scales comparable to the orbital period. Thus, in the case of the gravitational potential the focus is on the effect of even-degree zonal harmonics, which are known to cause secular effects. Odd-degree zonal harmonics may also be important because they originate long-period effects, whereas the effects of tesseral harmonics in general average out to zero. The latter, however, can have an important effect in resonant orbits, as in the case of geostationary satellites (1 to 1 resonance) or GPS and Molniya orbits (2 to 1 resonance).

The importance of each perturbation acting on an earth's satellite depends on the orbit's characteristics, and fundamentally on the altitude of the satellite, but also on its mean motion. Thus, for instance the atmospheric drag, which can have an important impact in the lower orbits, may be taken as a higher order effect for altitudes above, say, 800 km over the earth's surface, and is almost negligible above 2000 km.

A sketch of the order of different perturbations when compared to the Keplerian attraction is presented in Fig. 1 based on approximate formulas borrowed from [1, p. 114]. As illustrated in the figure, the non-centralities of the Geopotential have the most important effect in those parts of the orbit that are below the geosynchronous distance, where the $J_2$ contribution is a first order effect and other harmonics cause second order effects. To the contrary, in those parts of the orbit that are farther than the geosynchronous distance the gravitational pull of the moon is the most important perturbation, whereas

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that of the sun is of second order when compared to the disturbing effect of the moon, and perturbations due to $J_2$ and solar radiation pressure (SRP) are of third order.

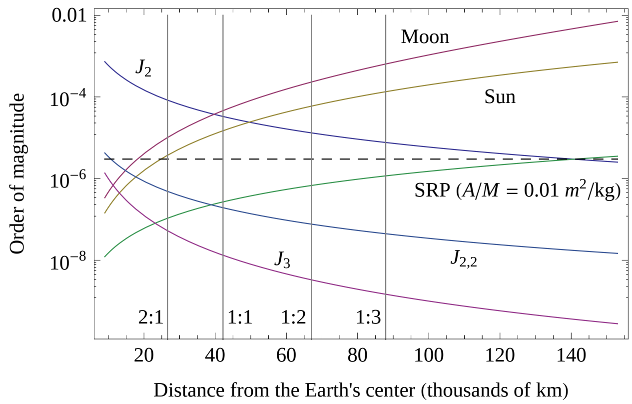


**Fig. 1**. Perturbation order relative to the Keplerian attraction.

Finally, we recall that integration of osculating elements is properly done only in the True of Date system [2]. For this reason Chapront's solar and lunar ephemeris are used [3, 4], which are directly referred to the mean of date thus including the effect of equinoctial precession.

In the case of a highly elliptic orbit (HEO) the distance of the satellite to the earth's center of mass varies notably along the orbit, a fact that makes particularly difficult to establish the main perturbation over which to set up the correct perturbation arrangement. This issue is aggravated by the importance of the gravitational pull of the moon on high altitude orbits, which requires taking higher degrees of the Legendre polynomial expansion of the third-body disturbing function into account. Since this expansion converges slowly, the size of the multivariate Fourier series representing the moon perturbation will soon grow enormously. Besides, for operational reasons, different HEOs may be synchronized with the earth rotation, and hence, being notably affected by resonance effects. Also, a HEO satellite will spend most of the time in the apogee region, where the solar radiation pressure may have an observable effect in the long-term, yet depending on the physical characteristics of the satellite. Finally, the perigee of usual geostationary transfer orbits (GTO) will enter repeatedly the atmosphere, although only for short periods.

Therefore, a long-term orbit propagator for HEO aiming at describing at least qualitatively the orbit evolution must consider the following perturbations:

- the effects of the main zonal harmonics of the Geopotential as well as second order effects of $J_2$;
- tesseral effects for the most common resonances. In particular, the 2:1 affecting Molnya orbits, or super synchronous resonances affecting space telescope orbits;
- lunisolar perturbations in the mass-point approximation, including at least the effects of the first few terms of the Legendre polynomials expansion of the third-body perturbation for the moon, and at least the effect of the polynomial of the second degree for the sun;
- solar radiation pressure effects;
- and the circularizing effects of atmospheric drag affecting the orbit semi-major axis and eccentricity.

In what respects to the ephemeris of the sun and moon needed for evaluating the corresponding disturbing potentials, the precision provided by simplified analytical formulas should be enough under the accuracy of a perturbation theory. For this reason truncated series from [5] of Chapront's solar and lunar ephemeris are enough at the precision of the mean elements propagation and notably speed the computations.

## 3. HAMILTONIAN ARRANGEMENT

The averaged Hamiltonian is constructed by the Lie transforms technique [6], including the time dependence in the solution of the homological equation.

Thus, the Hamiltonian is arranged as a power series in a small parameter $\epsilon$, viz. $\mathcal{H} = \sum_{m\geq 0}(\epsilon^m/m!)\mathcal{H}_{m,0}$, with

$$\begin{aligned}
\mathcal{H}_{0,0} &= -\mu/(2a)\\
\mathcal{H}_{1,0} &= -(\mu/r)(R_\oplus/r)^2 C_{2,0}P_2(\sin\varphi)\\
\frac{1}{2}\mathcal{H}_{2,0} &= -(\mu/r)\sum_{m>2}(R_\oplus/r)^m C_{m,0}P_m(\sin\varphi) + \mathcal{T} + \mathcal{V}_{☾} + \mathcal{V}_\odot
\end{aligned}$$

where $\mu$ is the earth's gravitational parameter, $(r,\varphi,\lambda)$ are spherical coordinates and $a$ is the semi-major axis of the satellite's orbit, $R_\oplus$ is the earth's equatorial radius, $P_m$ are Legendre polynomials, and $C_{m,0}$ are zonal harmonic coefficients. The tesseral disturbing function is

$$\mathcal{T} = -\frac{\mu}{r}\sum_{m\geq 2}\frac{R_\oplus^m}{r^m}\sum_{n=1}^{m}P_{m,n}(\sin\varphi)[C_{m,n}\cos n\lambda + S_{m,n}\sin n\lambda]$$

in which $P_{m,n}$ are associated Legendre polynomials while $C_{n,m}$ and $S_{n,m}$, $m\neq 0$, are non-zonal harmonic coefficients. Under the assumption of point masses,

$$\mathcal{V}_\star = -(\mu_\star/r_\star)\left(r_\star/||\boldsymbol{r}-\boldsymbol{r}_\star|| - \boldsymbol{r}\cdot\boldsymbol{r}_\star/r_\star^2\right) \qquad (1)$$

where $\mathcal{V}_\star \equiv \mathcal{V}_{☾}$ for the moon, and $\mathcal{V}_\star \equiv \mathcal{V}_\odot$ in the case of the sun. The disturbing body is far away from the perturbed body when dealing with perturbed Keplerian motion. Then Eq. (1) can be expanded in power series of the ratio $r/r_\star$

$$\mathcal{V}_\star = -\beta\,(n_\star^2 a_\star^3/r_\star)\sum_{m\geq 2}(r/r_\star)^m P_m(\cos\psi_\star), \qquad (2)$$

where $\beta = m_\star/(m_\star + m)$ and

$$\cos\psi_\star = (xx_\star + yy_\star + zz_\star)/(rr_\star) \qquad (3)$$

The semi-analytic theory only considers $P_2$ in Eq. (2) for the sun potential, whereas $P_2$–$P_6$ are taken for the moon.

The Lie transforms averaging is only developed up to the second order in the small parameter, so there is no coupling between the different terms of the disturbing function. Hence, the generating function of the averaged Hamiltonian can be split into different terms which are simply added at the end.

### 3.1. Time dependency

Orbits with large semi-major axis will experience high perturbations from the moon's gravity, and may be better described under the three-body problem model. However, since our approach is based on perturbed Keplerian motion, we still take as the zero order Hamiltonian the Keplerian term.

Because the sun and moon ephemeris are known functions of time, the perturbed problem remains of three degrees of freedom, but the Lie derivative must take the time dependency into account, viz. $\mathcal{L}_0(\mathcal{W}) \equiv \{H_{0,0}; \mathcal{W}\} + \partial\mathcal{W}/\partial t = n\,\partial\mathcal{W}/\partial\ell + \partial\mathcal{W}/\partial t$.

Dealing explicitly with time can be avoided by moving to the extended phase space. Then, assuming that the semi-major axis, eccentricity, and inclination of the third-body orbits, remain constant

$$\mathcal{L}_0 = n\,\frac{\partial\mathcal{W}}{\partial\ell} + n_{☾}\frac{\partial\mathcal{W}}{\partial\ell_{☾}} + \dot{g}_{☾}\frac{\partial\mathcal{W}}{\partial g_{☾}} + \dot{h}_{☾}\frac{\partial\mathcal{W}}{\partial h_{☾}} + n_{\odot}\frac{\partial\mathcal{W}}{\partial\ell_{\odot}} + \ldots$$

which, in view of the period of the lunar perigee of 8.85 years (direct motion) and of the lunar node of 18.6 years (retrograde motion), can be safely approximated by

$$\mathcal{L}_0 \approx n\,\partial\mathcal{W}/\partial\ell + n_{☾}\partial\mathcal{W}/\partial\ell_{☾} + n_{\odot}\partial\mathcal{W}/\partial\ell_{\odot} \qquad (4)$$

Note, however, that in the present stage, the theory only deals with mean elements and corresponding homological equations does not need to be solved. Future versions of the theory may take the short-period corrections due to third-body perturbations into account.

### 3.2. Third body direction

For the semi-analytic theory, the time dependency will manifest only in the short-period corrections, which are derived from the generating function $\mathcal{W}$. In view of the form of the Lie derivative in Eq. (4), the directions of both the sun and the moon must be expressed as functions of the sun and moon mean anomaly, respectively. For the sun, we use

$$\boldsymbol{r}_{\odot} = R_1(-\varepsilon)\,R_3(-\lambda_{\odot})\,R_2(\beta_{\odot})\,(r_{\odot},0,0)^{\tau} \qquad (5)$$

where $\tau$ means transposition, $\varepsilon$ is the mean obliquity of the ecliptic, and $r_{\odot}$ is the radius of the sun, and $\beta_{\odot}$ and $\lambda_{\odot}$ are the ecliptic latitude and longitude of the sun, respectively. The sun's latitude can be neglected in view of it never exceeds 1.2 arc seconds when referred to the ecliptic of the date.

In the case of the moon, because the lunar inclination to the equator is not constant, the orbit is rather referred to the mean equator and equinox of the date by means of

$$\boldsymbol{r}_{☾} = R_1(-\varepsilon)\,R_3(-\Omega_{☾})\,R_1(-J)\,R_3(-\theta_{☾})\,(r_{☾},0,0)^{\tau}$$

where $\theta_{☾}$ is the argument of the latitude of the moon, $J \approx 5^{\circ}\!.15$ is the inclination of the moon orbit over the ecliptic, which is affected of periodic oscillations whose period slightly shorter than half a year because the retrograde motion of the moon's line of nodes, and $\Omega_{☾}$ is the longitude of the ascending node of the moon orbit with respect to the ecliptic measured from the mean equinox of date. Solar and lunar ephemerides are taken from the low precision formulas in [5].

Besides, for orbital applications, instead of using spherical coordinates the satellite's radius vector is expressed in orbital elements; This can be done by first replacing the spherical coordinates by Cartesian ones: $\sin\varphi = z/r$, $\sin\lambda = y/q$, $\cos\lambda = x/\rho$, where $q = \sqrt{x^2+y^2}$. Then, the orbital and inertial frame are related by means of simple rotations $R_i(\alpha)$:

$$(x,y,z)^{\tau} = R_3(-\Omega)\,R_1(-I)\,R_3(-\theta)\,(r,0,0)^{\tau} \qquad (6)$$

where $\theta = f+\omega$, $r = p/(1+e\cos f)$, and $(a,e,I,\Omega,\omega,M)$ are traditional orbital elements.

## 4. AVERAGING ZONAL TERMS

The zonal part of the Hamiltonian is first transformed. Because of the actual values of the zonal coefficients of the earth, the old Hamiltonian is arranged in the form

$$\mathcal{H}_{0,0} = -\mu/(2a) \qquad (7)$$

$$\mathcal{H}_{1,0} = -(\mu/r)\,(R_{\oplus}/r)^2\,C_{2,0}P_2(\sin I\sin\theta) \qquad (8)$$

$$\mathcal{H}_{2,0} = -2(\mu/r)\sum_{m>2}(R_{\oplus}/r)^2\,C_{m,0}P_m(\sin I\sin\theta) \qquad (9)$$

where all the symbols, viz. $a$, $r$, $I$, and $\theta$, are assumed to be functions of some set of canonical variables. In particular, the averaging is carried out based on the canonical set of Delaunay variables, which is made of the coordinates $\ell = M$, $g = \omega$, $h = \Omega$, and their conjugate momenta $L = \sqrt{\mu a}$, $G = L\eta$, $H = G\cos I$, respectively. These variables are the action-angle variables of the Keplerian motion.

The homological equation is $-n(\partial W_m/\partial\ell) + \widetilde{\mathcal{H}}_{0,m} = \mathcal{H}_{0,m}$, and hence terms of the generating function are computed from quadratures.

### 4.1. Elimination of the parallax

It is known that the removal of short-period terms from the Hamiltonian is facilitated to a large extent by a preprocessing of the original Hamiltonian in order to remove parallactic terms [7]. That is, by first applying the parallactic identity

$$\frac{1}{r^m} = \frac{1}{r^2}\frac{1}{r^{m-2}} = \frac{1}{r^2}\frac{(1+e\cos f)^{m-2}}{p^{m-2}}, \quad m>2, \qquad (10)$$

and then selecting $H_{0,m}$ by removing all the trigonometric terms of the expansion of $H_{m,0}$ as a Fourier series that explicitly contain the true anomaly [8, 9].

The 1st step of Deprit's recurrence is $\widetilde{\mathcal{H}}_{0,1} \equiv \mathcal{H}_{1,0}$, where

$$\mathcal{H}_{1,0} = C_{2,0}(R_\oplus/r)^2 n^2 p^2/(8\eta^6)\big\{(4-6s^2)(1+e\cos f) \\ +3\left[e\cos(f+2\omega)+2\cos(2f+2\omega)+e\cos(3f+2\omega)\right]s^2\big\}$$

Then, the new Hamiltonian term $\mathcal{H}_{0,1}$ is selected as

$$\mathcal{H}_{0,1} = C_{2,0}(R_\oplus/r)^2 n^2 p^2 (1/\eta^6)(1/4)\left(2-3s^2\right) \quad (11)$$

and the generating function term $W_1$ must be solved from the homological equation $-n(\partial W_1/\partial\ell)+\widetilde{\mathcal{H}}_{0,1} = \mathcal{H}_{0,1}$. That is,

$$W_1 = (1/n)\int\Big\{C_{2,0}(R_\oplus/r)^2 n^2 p^2/(8\eta^6) \quad (12) \\ \times\Big[\left(4-6s^2\right)e\cos f + s^2\sum_{j=1,3} jE_{1,j}\cos(jf+2\omega)\Big]\Big\}\mathrm{d}\ell$$

in which $E_{1,1}=3e$, $E_{1,2}=3$, $E_{1,3}=e$. Equation (12) is solved in closed form by recalling the differential relation

$$\mathrm{d}M = (r/p)^2\eta^3\,\mathrm{d}f, \quad (13)$$

based on the preservation of the angular momentum of the Keplerian motion. We get

$$W_1 = k + nR_\oplus^2\frac{C_{2,0}}{8\eta^3}\left[(4-6s^2)e\sin f + s^2\sum_{j=1}^{3}E_{1,j}\sin(jf+2\omega)\right]$$

where $k$ is an arbitrary function independent of $\ell$.

To avoid the appearance of hidden long-period terms in $W_1$, $k$ is chosen to guarantee that $\frac{1}{2\pi}\int_0^{2\pi} W_1\,\mathrm{d}M = 0$. Using, again, Eq. (13) to compute the quadrature, we get

$$k = (1/8)nR_\oplus^2 C_{2,0}\eta^{-3}(1+2\eta)(1-\eta)(1+\eta)^{-1}s^2\sin 2\omega.$$

Therefore, calling $E_{1,0} = (1+2\eta)(1-\eta)/(1+\eta)$

$$W_1 = nR_\oplus^2\frac{C_{2,0}}{8\eta^3}\left[(4-6s^2)e\sin f + s^2\sum_{j=0,3}E_{1,j}\sin(jf+2\omega)\right]$$

At order 2, Deprit's recurrence gives: $\mathcal{H}_{0,2} = \{\mathcal{H}_{0,1}, W_1\} + \mathcal{H}_{1,1}$, and $\mathcal{H}_{1,1} = \{\mathcal{H}_{0,0}, W_2\} + \{\mathcal{H}_{1,0}, W_1\} + \mathcal{H}_{2,0}$. Therefore, the homological equation is $-n(\partial W_2/\partial\ell)+\widetilde{\mathcal{H}}_{0,2} = \mathcal{H}_{0,2}$, where $\widetilde{\mathcal{H}}_{0,2} = \{\mathcal{H}_{0,1}, W_1\} + \{\mathcal{H}_{1,0}, W_1\} + \mathcal{H}_{2,0}$.

After performing the required operations, parallactic terms are removed from $\widetilde{\mathcal{H}}_{0,2}$ using Eq. (10). Then, $\widetilde{\mathcal{H}}_{0,2}$ is written in the form of a Poisson series, and $\mathcal{H}_{0,2}$ is chosen by removing the trigonometric terms of $\widetilde{\mathcal{H}}_{0,2}$ that explicitly depend on the true anomaly, to give:

$$\mathcal{H}_{0,2} = 2(\mu/p)(p/r)^2\sum_{i=3,10}J_i^* - (\mu/p)(p/r)^2 C_{2,0}^2(R_\oplus/p)^4 \\ \times\Big\{\frac{5}{4} - \frac{21}{8}s^2 + \frac{21}{16}s^4 + \frac{3}{8}\Big(c^2 - \frac{5}{8}s^4\Big)e^2 \\ -\frac{3}{8}\Big[\frac{15}{2} - \frac{35}{4}s^2 - \left(4-5s^2\right)\frac{\eta^2}{(1+\eta)^2}\Big]e^2 s^2\cos 2\omega\Big\}$$

with

$$J_i^* = C_{i,0}\frac{R_\oplus^i}{p^i}\sum_{j=0}^{\lfloor i/2\rfloor-1} e^l Q_{i,l}s^l B_{i,l}\,\mathbb{i}^m\exp(\mathbb{i}\,l\omega) \quad (14)$$

where $l = 2j+m$, $\lfloor\ \rfloor$ notes an integer division, $m \equiv i \bmod 2$, and $\mathbb{i} = (-1)^{1/2}$. The eccentricity polynomials $Q_{i,j}$ and inclination polynomials $B_{i,j}$ are given in Tables 1, 2, and 3, respectively. Note that Eq. (14) applies also to $i=2$; indeed $\mathcal{H}_{0,1} = (\mu/p)(p/r)^2 J_2^*$.

Finally, the simplified Hamiltonian is obtained by replacing the old variables by the new ones in all the $\mathcal{H}_{0,m}$ terms. That is, the new Hamiltonian is obtained by assuming that all symbols that appear in the $\mathcal{H}_{0,m}$ terms are functions of the new (prime) Delaunay variables.

### 4.2. Delaunay normalization

After the preparatory simplification, it is trivial to remove the remaining short-period terms from the simplified Hamiltonian $\mathcal{H}' = \sum(\epsilon^m/m!)\mathcal{H}'_{m,0}$, where $\mathcal{H}'_{0,0}$ is the same as $\mathcal{H}_{0,0}$, $\mathcal{H}'_{1,0}$ is the same as $\mathcal{H}_{0,1}$ in Eq. (11), and, $\mathcal{H}'_{2,0}$ is the same as $\mathcal{H}_{0,2}$, but all of them expressed in the new, prime variables.

The new Hamiltonian term $\mathcal{H}'_{0,1}$ is chosen by removing the short-period terms from $\mathcal{H}'_{1,0}$, namely

$$\mathcal{H}'_{0,1} = \frac{1}{2\pi}\int_0^{2\pi}\mathcal{H}'_{1,0}\mathrm{d}M = \frac{\mu}{p}C_{2,0}\frac{R_\oplus^2}{p^2}\eta^3\left(\frac{1}{2}-\frac{3}{4}s^2\right)$$

which was trivially solved using Eq. (13).

The first term of the new generating function is solved by quadrature from the homological equation from which

$$W'_1 = \frac{1}{n}\Big[-\mathcal{H}'_{0,1}\ell + \int\mathcal{H}'_{1,0}(r/p)^2\eta^3\,\mathrm{d}f\Big] = \frac{\phi}{n}\mathcal{H}'_{0,1},$$

where $\phi = f - M$ is the equation of the center. Since $\phi$ is made only of short-period terms, there is no need of introducing additional integration constants.

At the second order, the computable terms of the homological equation are $\widetilde{\mathcal{H}}'_{0,2} = \{\mathcal{H}'_{0,1}, W'_1\} + \{\mathcal{H}'_{1,0}, W'_1\} + \mathcal{H}'_{2,0}$, from which expression the new Hamiltonian term $\mathcal{H}'_{0,2}$ is chosen by removing the short-period terms.

After performed the required operations and replacing prime variables by new, double prime variables in all the $\mathcal{H}'_{0,m}$ terms, we get the averaged Hamiltonian $\mathcal{H}'' = \mathcal{H}'_{0,1} + (1/2)\mathcal{H}'_{0,2}$, viz.

$$\mathcal{H}'' = \frac{\mu}{p}\eta^3\sum_{i=2,10}J_i^* + \frac{\mu}{p}\eta^3 C_{2,0}^2(R_\oplus/p)^4(3/16) \quad (15)$$

$$\times\left\{(1-5c^2)c^2-\left(\frac{1}{3}+s^2-\frac{17}{8}s^4\right)e^2-(1-3c^2)^2\right.$$
$$\left.\times\frac{1}{2}\eta-\left[\frac{5}{4}(1-7c^2)-\frac{(1-5c^2)\eta^2}{(1+\eta)^2}\right]e^2s^2\cos 2\omega\right\},$$

## 5. THIRD-BODY AVERAGING

Now, the perturbation Hamiltonian is arranged

$$\mathcal{H}_{0,0}=-\mu/(2a),\qquad \mathcal{H}_{1,0}=0,\qquad \mathcal{H}_{2,0}=2\mathcal{V}_{☾}+2\mathcal{V}_{\odot}$$

where the sun and moon potentials are computed from Eq. (2).

Since the maximum power of $\chi$ in $P_m(\chi)$ is $\chi^m$, in view of Eqs. (2) and (3), we check that the satellite's radius $r$ appears now in numerators, contrary to the geopotential case. Therefore, the closed form theory is approached now using the eccentric anomaly $u$ instead of the true one $f$. Hence, $r=a(1-e\cos u)$, and the Cartesian coordinates of the satellite are obtained from Eq. (6) using the known relations on the ellipse: $r\sin f=a\eta\sin u$, $r\cos f=a(\cos u-e)$.

Because $\mathcal{H}_{1,0}\equiv 0$ we choose $\mathcal{H}_{0,1}=0$ and trivially find $W_1=0$. Then, the second order of the homological equation is $\mathcal{L}_0(W_2)+\widetilde{\mathcal{H}}_{0,2}=\mathcal{H}_{0,2}$, where, from Deprit's recurrence, $\widetilde{\mathcal{H}}_{0,2}=\mathcal{H}_{2,0}$. The new Hamiltonian term $\mathcal{H}_{0,2}$ is chosen by removing the short-period terms in $\mathcal{H}_{2,0}$. Again, this is done by closed-form averaging $\mathcal{H}_{0,2}=\frac{1}{2\pi}\int_0^{2\pi}\mathcal{H}_{2,0}(r/a)\,\mathrm{d}u$ where we used the differential relation

$$\mathrm{d}M=(1-e\cos u)\,\mathrm{d}u=(r/a)\,\mathrm{d}u \tag{16}$$

which is obtained from Kepler equation.

The averaging produces the long-term Hamiltonian [10]

$$\mathcal{H}_{0,2}=2(na)^2\beta^*(a_\star/r_\star)^3(n_\star/n)^2\sum_{m\geq 2}(a/r_\star)^{m-2}\,\Gamma_m$$

with the non-dimensional coefficients

$$\Gamma_m=\sum_{j=0}^{\lfloor m/2\rfloor}A_{m,j}\sum_{l=-m}^{m}P_{m,j,l}\left(S^\star_{m,l}\cos\alpha+T^\star_{m,l}\sin\alpha\right) \tag{17}$$

where $\alpha=(2j+k)\omega+l\Omega$, and $k=m \bmod 2$.

The eccentricity coefficients $A_{m,j}(e)$, the inclination ones $P_{m,j,l}(i)$, and the third-body direction coefficients $T^\star_{m,l}(u,v,w)$, $S^\star_{m,l}(u,v,w)$, are given in Tables 4, 5–9, and 10, respectively. They are valid for both the moon ($\star\equiv☾$) and the sun ($\star\equiv\odot$) by using, when required, the proper third-body direction vector $(u,v,w)\equiv(u^\star,v^\star,w^\star)$.

## 6. TESSERAL RESONANCES

The tesseral potential is no longer symmetric with respect to the earth's rotation axis. Therefore, longitude dependent terms will explicitly depend on time in the inertial frame. To avoid the explicit appearance of time in the Hamiltonian, we move to a rotating frame with the same frequency as the rotation of the earth. The argument of the node in the rotating frame is $h=\Omega-n_\oplus t$, where $n_\oplus$ is the earth's rotation rate, and, to preserve the Hamiltonian character, we further introduce the Coriolis term $-n_\oplus H$. It is then simple to check that $H=\Theta\cos I$ still remains as the conjugate momentum to $h$.

Thus, the tesseral Hamiltonian is arranged as

$$\mathcal{H}_{0,0}=-\mu/(2a)-n_\oplus\Theta\cos I,\quad \mathcal{H}_{1,0}=0,\quad \mathcal{H}_{2,0}=2\mathcal{T},$$

where, now, $\mathcal{H}_{0,0}$ is the Keplerian in the rotating frame. Hence, the Lie derivative reads $\{\mathcal{H}_{0,0};W\}=-n\partial W/\partial\ell+n_\oplus\partial W/\partial h$, and the solution of the homological equation will introduce denominators of the type $(in-jn_\oplus)$, with $i$ and $j$ integer. Therefore, resonances between the rotation rate of the node in the rotating frame and mean motion of the satellite $i/j=n_\oplus/n$ introduce the problem of small divisors.

In fact, resonant tesseral terms introduce long-period terms in the semi-major axis that may be not negligible even at the limited precision of a long-term propagation. Therefore, these terms must remain in the long-term Hamiltonian. Furthermore, these terms must be traced directly in the mean, contrary to true, anomaly to avoid leaving short-period terms in the Hamiltonian, which will destroy the performance of the semi-analytical integration. Therefore, trigonometric functions of the true or the eccentric anomaly must be expanded as Fourier series in the mean anomaly whose coefficients are power series in the eccentricity.

After the short-period terms have been removed from the tesseral Hamiltonian, we come back to the inertial frame by dropping the Coriolis term and replacing $h$ by the RAAN, in this way explicitly showing the time into resonant terms of the long-term Hamiltonian.

From Kaula expansions [11], we find that the main terms of the Geopotential that are affected by the 2:1 tesseral resonance are $\mathcal{R}_{2:1}=-(\mu/a)(R_\oplus/a)^2(3/4)R_{2:1}$, with

$$\begin{aligned}R_{2:1}=&F_{2,2,0}G_{2,0,-1}\left[C_{2,2}\cos(\alpha+2\omega)+S_{2,2}\sin(\alpha+2\omega)\right]\\&+F_{2,2,1}G_{2,1,1}\left(C_{2,2}\cos\alpha+S_{2,2}\sin\alpha\right)\\&+F_{2,2,2}G_{2,2,3}\left[C_{2,2}\cos(\alpha+2\omega)+S_{2,2}\sin(\alpha-2\omega)\right]\end{aligned}$$

where $\alpha=2(\Omega-n_\oplus t)+M$ is the resonant angle,

$$F_{2,2,0}=(1+c)^2,\quad F_{2,2,1}=2s^2,\quad F_{2,2,2}=(1-c)^2 \tag{18}$$

and, up to $\mathcal{O}(e^{16})$,

$$\begin{aligned}G_{2,0,-1}=&-\frac{1}{2}e+\frac{1}{16}e^3-\frac{5}{384}e^5-\frac{143}{18432}e^7-\frac{9097e^9}{1474560}\\&-\frac{878959e^{11}}{176947200}-\frac{121671181e^{13}}{29727129600}-\frac{4582504819e^{15}}{1331775406080}\end{aligned}$$

$$\begin{aligned}G_{2,1,1}=&\frac{3}{2}e+\frac{27}{16}e^3+\frac{261}{128}e^5+\frac{14309}{6144}e^7+\frac{423907}{163840}e^9\\&+\frac{55489483}{19660800}e^{11}+\frac{30116927341}{9909043200}e^{13}+\frac{2398598468863}{739875225600}e^{15}\end{aligned}$$

$$G_{2,2,3} = \frac{1}{48}e^3 + \frac{11}{768}e^5 + \frac{313}{30720}e^7 + \frac{3355}{442368}e^9 + \frac{1459489e^{11}}{247726080} + \frac{187662659e^{13}}{39636172800} + \frac{33454202329e^{15}}{8561413324800}$$

Other 2:1-resonant terms can be found in [12]. For the 1:1 resonance, $\mathcal{R}_{1:1} = -(\mu/a)(R_\oplus/a)^2(3/4)R_{1:1}$, with

$$\begin{aligned} R_{1:1} &= [C_{2,2}\cos(2\alpha+2\omega) + S_{2,2}\sin(2\alpha+2\omega)]\, F_{2,2,0} \\ &\times G_{2,0,0} + [C_{2,2}\cos 2\alpha + S_{2,2}\sin 2\alpha]\, F_{2,2,1}G_{2,1,2} \\ &+ F_{2,1,0}G_{2,0,-1}\,[C_{2,1}\sin(\alpha+2\omega) - S_{2,1}\cos(\alpha+2\omega)] \\ &+ F_{2,1,1}G_{2,1,1}\,(C_{2,1}\sin\alpha - S_{2,1}\cos\alpha) \\ &+ F_{2,1,2}G_{2,2,3}\,[-S_{2,1}\cos(\alpha-2\omega) + C_{2,1}\sin(\alpha-2\omega)] \end{aligned}$$

where now $\alpha = \Omega - n_\oplus t + M$.

For the earth, $C_{2,1} = \mathcal{O}(10^{-10})$ and $S_{2,1} = \mathcal{O}(10^{-9})$; hence corresponding terms are commonly neglected from the resonant tesseral potential $\mathcal{R}_{1\times 1}$. Therefore, the needed inclination polynomials are only $F_{2,2,0}$ and $F_{2,2,1}$, which were already given in Eq. (18), whereas the required eccentricity functions, up to $\mathcal{O}(e^{16})$, are

$$\begin{aligned} G_{2,0,0} &= 1 - \frac{5}{2}e^2 + \frac{13}{16}e^4 - \frac{35}{288}e^6 - \frac{5}{576}e^8 - \frac{49}{3600}e^{10} \\ &- \frac{3725}{331776}e^{12} - \frac{7767869}{812851200}e^{14} - \frac{5345003}{650280960}e^{16} \\ G_{2,1,2} &= \frac{9}{4}e^2 + \frac{7}{4}e^4 + \frac{141}{64}e^6 + \frac{197}{80}e^8 + \frac{62401}{23040}e^{10} \\ &+ \frac{262841}{89600}e^{12} + \frac{9010761}{2867200}e^{14} + \frac{8142135359}{2438553600}e^{16} \end{aligned}$$

## 7. GENERALIZED FORCES

The evolution equations are completed adding the averaged effects of the generalized forces to the Hamilton equations.

### 7.1. SRP

In the cannonball approximation, the perturbing acceleration caused by solar-radiation pressure, is always in the opposite direction of the unit vector of the sun $\boldsymbol{\alpha}_{\rm srp} = -F_{\rm srp}\boldsymbol{i}_\odot$. If, besides, it is assumed that the parallax of the sun is negligible, the solar flux is constant along the orbit of the satellite, and there is no re-radiation from the earth's surface [13], then $F_{\rm srp} = (1+\beta)P_\odot\,(a_\odot/r_\odot)^2(A/m)$ where $\beta$ is the index of reflection ($0 < \beta < 1$), $A/m$ is the area-to-mass ratio of the spacecraft, $a_\odot$ is the semi-major axis of the sun's orbit around earth, $r_\odot$ is the radius of the sun, and $P_\odot \approx 4.56\times 10^{-6}\,\mathrm{N/m^2}$ is the SRP constant at one AU [1, p. 77].

The components of $\boldsymbol{i}_\odot$ in the radial, tangent, and normal directions, respectively, are obtained by simple rotations

$$\boldsymbol{i}_\odot = R_3(\theta)\,R_1(I)\,R_3(\Omega)\,R_1(-\varepsilon)\,R_3(-\lambda_\odot)\,(1,0,0)^\tau$$

where $\lambda_\odot$ is the ecliptic longitude of the sun, and $\varepsilon$ is the obliquity of the ecliptic.

Then, calling $F = -F_{\rm srp}/\mu$, Kozai's analytical expressions for perturbations due to SRP [13] are easily recovered from the usual Gauss equations. After averaging over the mean anomaly, which is done in closed form based on Eq. (16), we get $\overline{\mathrm{d}a/\mathrm{d}t} = 0$, and

$$\begin{aligned} \overline{\frac{\mathrm{d}e}{\mathrm{d}t}} &= (3/4)na^2F\eta\big\{\sin\omega\big[(\cos\varepsilon-1)\cos(\lambda_\odot+\Omega) \\ &-(\cos\varepsilon+1)\cos(\lambda_\odot-\Omega)\big] + \cos\omega\big[2s\sin\varepsilon \\ &\times\sin\lambda_\odot + c(\cos\varepsilon+1)\sin(\lambda_\odot-\Omega) \\ &+c(\cos\varepsilon-1)\sin(\lambda_\odot+\Omega)\big]\big\} \\ \overline{\frac{\mathrm{d}I}{\mathrm{d}t}} &= (3/4)na^2(e/\eta)F\cos\omega\big[s(\cos\varepsilon+1)\sin(\lambda_\odot-\Omega) \\ &-2c\sin\varepsilon\sin\lambda_\odot + s(\cos\varepsilon-1)\sin(\lambda_\odot+\Omega)\big] \\ \overline{\frac{\mathrm{d}\Omega}{\mathrm{d}t}} &= \frac{3}{4}na^2\frac{e}{\eta}\frac{1}{s}F\sin\omega\big[s(\cos\varepsilon+1)\sin(\lambda_\odot-\Omega) \\ &-2c\sin\varepsilon\sin\lambda_\odot + s(\cos\varepsilon-1)\sin(\lambda_\odot+\Omega)\big] \\ \overline{\frac{\mathrm{d}\omega}{\mathrm{d}t}} &= -\frac{3}{4}na^2\frac{F}{e\eta}\big\{\sin\omega\big[(\cos\varepsilon+1)\sin(\lambda_\odot-\Omega) \\ &\times c + 2\left(s - e^2/s\right)\sin\varepsilon\sin\lambda_\odot + c(\cos\varepsilon-1) \\ &\times\sin(\lambda_\odot+\Omega)\big] + \eta^2\cos\omega\big[(\cos\varepsilon+1) \\ &\times\cos(\lambda_\odot-\Omega) + (1-\cos\varepsilon)\cos(\lambda_\odot+\Omega)\big]\big\} \\ \overline{\frac{\mathrm{d}M}{\mathrm{d}t}} &= n + \frac{3}{4}na^2\frac{e^2+1}{e}F\big\{\sin\omega\big[c(\cos\varepsilon+1) \\ &\times\sin(\lambda_\odot-\Omega) + c(\cos\varepsilon-1)\sin(\lambda_\odot+\Omega) \\ &+2s\sin\varepsilon\sin\lambda_\odot\big] + \cos\omega\big[(\cos\varepsilon+1) \\ &\times\cos(\lambda_\odot-\Omega) + (1-\cos\varepsilon)\cos(\lambda_\odot+\Omega)\big]\big\} \end{aligned}$$

### 7.2. Atmospheric drag: Averaged effects

Predicting the atmospheric behavior for the accurate evaluation of drag effects seems naive for the long-term scales of interest in this study. However, the atmospheric drag may dominate over all other perturbations in the case of orbits with low perigee heights, even to the extent of forcing the satellite's de-orbit.

The magnitude of the drag force depends on the local density of the atmosphere $\rho$ and the cross-sectional area $A$ of the spacecraft in the direction of motion. The drag force per unit of mass $m$ is $\boldsymbol{\alpha}_{\rm drag} = -\frac{1}{2}n_{\rm d}\boldsymbol{V}$, where $\boldsymbol{V}$ is velocity of the spacecraft relative to the atmosphere, of modulus $V$, we abbreviated

$$n_{\rm d} = \rho BV > 0, \qquad (19)$$

and $B = (A/m)C_{\rm drag}$, is the so-called ballistic coefficient, with the dimensionless drag coefficient $C_{\rm drag}$ ranging from 1.5–3.0 for a typical satellite. Note that $n_{\rm d} \equiv n_{\rm d}(t)$.

A reasonable approximation of the relative velocity is obtained with the assumption that the atmosphere co-rotates with the earth. Then, from the derivative of a vector in a rotating frame, we get $\boldsymbol{V} = (\mathrm{d}\boldsymbol{r}/\mathrm{d}t) - \boldsymbol{\omega}_\oplus\times\boldsymbol{r}$. We further take $\boldsymbol{\omega}_\oplus = n_\oplus \boldsymbol{k}$, and compute its projections in the radial, normal, and bi-normal directions as

$$\boldsymbol{\omega}_\oplus = R_3(\theta)\, R_1(I)\, (0, 0, n_\oplus)^\tau$$

Then, the velocity components in the radial, normal, and bi-normal direction relative to a rotating atmosphere are

$$\boldsymbol{V} = (R, \Theta/r, 0)^\tau + r n_\oplus\, (0, -\cos I, \cos\theta \sin I)^\tau$$

where $R = \dot{r} = (\Theta/p) e \sin f$, and $\Theta = r^2\dot{\theta} = \sqrt{\mu p}$.

Models giving the atmospheric density are usually complex. Furthermore, since it depends on the solar flux which is not easily predictable, reliable predictions of the drag effect are not expected for long-term propagation. Hence, the aim is to show the effect that the atmospheric drag might have in the orbit, as opposite from a drag-free mode. Therefore, to speed evaluation of the semi-analytical propagator, we take advantage of the simplicity of the Harris-Priester density model, which is implemented with the modifications of [14].

Replacing $\boldsymbol{\alpha}_{\rm drag}$ into Gauss planetary equations, the long-term effects are computed after averaging the equations over the mean anomaly, viz.

$$\overline{\frac{\mathrm{d}a}{\mathrm{d}t}} = -\frac{a}{\eta^2}\frac{1}{2\pi}\int_0^{2\pi} n_{\rm d}\left(1 + 2e\cos f + e^2 - \frac{n_\oplus}{n}\eta^3 c\right)\mathrm{d}M$$

$$\overline{\frac{\mathrm{d}e}{\mathrm{d}t}} = \frac{1}{2\pi}\int_0^{2\pi} n_{\rm d}\left[\delta c\left(e + \cos f - \frac{e}{2}\sin^2 f\right) - e - \cos f\right]\mathrm{d}M$$

$$\overline{\frac{\mathrm{d}I}{\mathrm{d}t}} = -\frac{1}{2}s\frac{1}{2\pi}\int_0^{2\pi} n_{\rm d}\,\delta\cos^2\theta\,\mathrm{d}M$$

$$\overline{\frac{\mathrm{d}\Omega}{\mathrm{d}t}} = -\frac{1}{2}\frac{1}{2\pi}\int_0^{2\pi} n_{\rm d}\,\delta\sin\theta\cos\theta\,\mathrm{d}M$$

$$\overline{\frac{\mathrm{d}\omega}{\mathrm{d}t}} = -c\,\overline{\frac{\mathrm{d}\Omega}{\mathrm{d}t}} - \frac{1}{2\pi}\int_0^{2\pi}\frac{n_{\rm d}}{e}\sin f\left[1 - \delta c\left(1 + \frac{e}{2}\cos f\right)\right]\mathrm{d}M$$

$$\overline{\frac{\mathrm{d}M}{\mathrm{d}t}} = n + \frac{1}{2\pi}\int_0^{2\pi} n_{\rm d}\,\frac{e}{\eta}\frac{r}{a}\sin f\,\mathrm{d}M + \frac{1}{2\pi}\int_0^{2\pi} n_{\rm d}\frac{\eta}{e}\sin f\left[1 - \delta c\left(1 + \frac{e}{2}\cos f\right)\right]\mathrm{d}M$$

where the known relation $\sin u = (r/p)\eta\sin f$ in Eq. (20), and the abbreviation $\delta = (n_\oplus/n)(r/p)^2\eta^3$ have been used.

Both the relative velocity with respect to the rotating atmosphere $V$, and the atmospheric density $\rho$ are naturally expressed as a function of the true anomaly [14], then it happens that $n_{\rm d} \equiv n_{\rm d}(f)$ from the definition of $n_{\rm d}$ in Eq. (19). Hence, the quadratures above are conveniently integrated in $f$ rather than in $M$ using the differential relation in Eq. (13). Besides, due to the complex representation of the atmospheric density, these quadratures are evaluated numerically.

## 8. CONCLUSIONS

HEO propagation is a challenging problem because the different causes that have an effect in these kinds of orbits, whose relative influence may notably vary along the orbit. However, modern tools and methods allow to approach the problem by means of analytical methods. Indeed, using perturbation theory we succeeded in the implementation of a fast and efficient semi-analytical propagator which is able to capture the main frequencies of the HEO orbital motion over large spans. In particular, we used the Lie transforms method, which is standard these days in the construction of perturbation theories. This method is specifically designed for automatic computation by machine, and can be easily programmed with modern, commercial, general purpose software.

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| $k$ | $m=2$ | $m=4$ | $m=6$ | $m=8$ |
|---|---|---|---|---|
| 0 | 1 | $2+3e^2$ | $8+40e^2+15e^4$ | $3(16+168e^2+210e^4+35e^6)$ |
| 2 | | 1 | $6+3e^2$ | $48+80e^2+15e^4$ |
| 4 | | | 1 | $10+3e^2$ |
| 6 | | | | 1 |
| $k$ | | $m=5$ | $m=7$ | $m=9$ |
| 1 | | $4+3e^2$ | $3(8+20e^2+5e^4)$ | $3(64+336e^2+280e^4+35e^6)$ |
| 3 | | 1 | $8+3e^2$ | $5(16+20e^2+3e^4)$ |
| 5 | | | 1 | $12+3e^2$ |
| 7 | | | | 1 |
| $k$ | $m=10$ | | | |
| 0 | $3(128+2304e^2+6048e^4+3360e^6+315e^8)$ | | | |
| 2 | $15(32+112e^2+70e^4+7e^6)$ | | | |
| 4 | $15(8+8e^2+e^4)$ | | | |
| 6 | $14+3e^2$ | | | |
| 8 | 1 | | | |

**Table 1**. Eccentricity polynomials $Q_{m,k}$ in Eq. (14).

| $k$ | $m=1$ | $m=2$ | $m=3$ |
|---|---|---|---|
| 0 | $\frac{1}{4}\left(3c^2-1\right)$ | $-\frac{3}{128}\left(35c^4-30c^2+3\right)$ | $\frac{5}{2048}\left(231c^6-315c^4+105c^2-5\right)$ |
| 1 | | $-\frac{15}{64}\left(7c^2-1\right)$ | $\frac{175}{2048}\left(33c^4-18c^2+1\right)$ |
| 2 | | | $\frac{315}{4096}\left(11c^2-1\right)$ |
| | $m=4$ | | |
| 0 | $-\frac{35}{786432}\left(6435c^8-12012c^6+6930c^4-1260c^2+35\right)$ | | |
| 1 | $-\frac{2205}{131072}\left(143c^6-143c^4+33c^2-1\right)$ | | |
| 2 | $-\frac{4851}{131072}\left(65c^4-26c^2+1\right)$ | | |
| 3 | $-\frac{3003}{131072}\left(15c^2-1\right)$ | | |
| | $m=5$ | | |
| 0 | $\frac{21}{8388608}\left(46189c^{10}-109395c^8+90090c^6-30030c^4+3465c^2-63\right)$ | | |
| 1 | $\frac{693}{2097152}\left(4199c^8-6188c^6+2730c^4-364c^2+7\right)$ | | |
| 2 | $\frac{9009}{1048576}\left(323c^6-255c^4+45c^2-1\right)$ | | |
| 3 | $\frac{19305}{4194304}\left(323c^4-102c^2+3\right)$ | | |
| 4 | $\frac{109395}{16777216}\left(19c^2-1\right)$ | | |

**Table 2**. Even inclination polynomials $B_{2m,2k}$ in Eq. (14).

| $k$ | $m=1$ | $m=2$ | $m=3$ |
|---|---|---|---|
| 0 | $-\frac{3}{8}\left(5c^2-1\right)$ | $\frac{15}{128}\left(21c^4-14c^2+1\right)$ | $-\frac{35}{8192}\left(429c^6-495c^4+135c^2-5\right)$ |
| 1 | | $\frac{35}{256}\left(9c^2-1\right)$ | $-\frac{315}{16384}\left(143c^4-66c^2+3\right)$ |
| 2 | | | $-\frac{693}{16384}\left(13c^2-1\right)$ |
| | $m=4$ | | |
| 0 | $\frac{105}{262144}\left(2431c^8-4004c^6+2002c^4-308c^2+7\right)$ | | |
| 1 | $\frac{1617}{131072}\left(221c^6-195c^4+39c^2-1\right)$ | | |
| 2 | $\frac{3003}{131072}\left(85c^4-30c^2+1\right)$ | | |
| 3 | $\frac{6435}{524288}\left(17c^2-1\right)$ | | |

**Table 3**. Odd inclination polynomials $B_{2m+1,2k+1}$ in Eq. (14).

| $m$ | $j=0$ | 1 | 2 | 3 |
|---|---|---|---|---|
| 2 | $3(2+3e^2)$ | $-15e^2$ | | |
| 3 | $e(4+3e^2)$ | $e^3$ | | |
| 4 | $(8+40e^2+15e^4)$ | $e^2(2+e^2)$ | $e^4$ | |
| 5 | $e(8+20e^2+5e^4)$ | $e^3(8+3e^2)$ | $e^5$ | |
| 6 | $16+168e^2+210e^4+35e^6$ | $48e^2+80e^4+15e^6$ | $10e^4+3e^6$ | $e^6$ |

**Table 4**. Eccentricity polynomials $A_{m,j}$ in Eq. (17).

| | $P_{2,j,l}$ | | $P_{3,j,l}$ | |
|---|---|---|---|---|
| $l$ | $j=0$ | $j=1$ | $j=0$ | $j=1$ |
| 0 | $\frac{1}{48}(3c^2-1)$ | $-\frac{1}{16}s^2$ | $-\frac{15}{128}(5c^2-1)s$ | $-\frac{175}{128}s^3$ |
| $\pm1$ | $\frac{1}{8}cs$ | $\frac{1}{8}\chi s$ | $-\frac{15}{512}\chi(15c^2\mp10c-1)$ | $-\frac{525}{512}\chi s^2$ |
| $\pm2$ | $-\frac{1}{32}s^2$ | $\frac{1}{32}\chi^2$ | $\frac{75}{256}\chi(3c\mp1)s$ | $-\frac{525}{256}\chi^2 s$ |
| $\pm3$ | | | $\frac{75}{512}\chi s^2$ | $\frac{175}{512}\chi^3$ |

**Table 5**. Inclination polynomials $P_{m,j,l}$ in Eq. (17) ($\chi=c\pm1$).

| $l$ | $j=0$ | $j=1$ | $j=2$ |
|---|---|---|---|
| 0 | $-\frac{3}{4096}(35c^4-30c^2+3)$ | $-\frac{105}{1024}(7c^2-1)s^2$ | $-\frac{2205}{4096}s^4$ |
| $\pm1$ | $\frac{15}{1024}c(3-7c^2)s$ | $\frac{105}{512}\chi(14c^2\mp7c-1)s$ | $\frac{2205}{1024}\chi s^3$ |
| $\pm2$ | $\frac{15}{1024}(7c^2-1)s^2$ | $\frac{105}{256}\chi^2(7c^2\mp7c+1)$ | $\frac{2205}{1024}\chi^2 s^2$ |
| $\pm3$ | $\frac{105}{1024}cs^3$ | $\frac{735}{512}\chi^2(2c\mp1)s$ | $-\frac{2205}{1024}\chi^3 s$ |
| $\pm4$ | $-\frac{105}{2048}s^4$ | $-\frac{735}{512}\chi^2 s^2$ | $-\frac{2205}{2048}\chi^4$ |

**Table 6**. Inclination polynomials $P_{4,j,l}$ in Eq. (17) ($\chi=c\pm1$).

| $l$ | $j=0$ | $j=1$ | $j=2$ |
|---|---|---|---|
| 0 | $\frac{105}{8192}\left(21c^4-14c^2+1\right)s$ | $\frac{735}{16384}(9c^2-1)s^3$ | $\frac{14553}{16384}s^5$ |
| $\pm1$ | $\frac{105}{16384}\chi(105c^4\mp84c^3-42c^2\pm28c+1)$ | $\frac{2205}{32768}\chi(15c^2\mp6c-1)s^2$ | $\frac{72765}{32768}\chi s^4$ |
| $\pm2$ | $-\frac{735}{4096}\chi(15c^3\mp9c^2-3c\pm1)s$ | $\frac{2205}{8192}\chi^2(15c^2\mp12c+1)s$ | $\frac{72765}{8192}\chi^2 s^3$ |
| $\pm3$ | $\frac{735}{32768}\chi(15c^2\mp6c-1)s^2$ | $\frac{735}{65536}\chi^3(3c\mp1)(15c\mp13)$ | $\frac{72765}{65536}\chi^3 s^2$ |
| $\pm4$ | $-\frac{2205}{16384}\chi(5c\mp1)s^3$ | $-\frac{6615}{32768}\chi^3(5c\mp3)s$ | $\frac{72765}{32768}\chi^4 s$ |
| $\pm5$ | $\frac{2205}{32768}\chi s^4$ | $\frac{6615}{65536}\chi^3 s^2$ | $\frac{14553}{65536}\chi^5$ |

**Table 7**. Inclination polynomials $P_{5,j,l}$ in Eq. (17) ($\chi=c\pm1$).

| $l$ | $j=0$ | $j=2$ |
|---|---|---|
| 0 | $-\frac{5}{65536}(231c^6-315c^4+105c^2-5)$ | $-\frac{2079}{65536}(11c^2-1)s^4$ |
| $\pm1$ | $-\frac{105}{32768}c(33c^4-30c^2+5)s$ | $\frac{2079}{32768}\chi(33c^2\mp11c-2)s^3$ |
| $\pm2$ | $-\frac{525}{262144}(33c^4-18c^2+1)s^2$ | $-\frac{10395}{262144}\chi^2(33c^2\mp22c+1)s^2$ |
| $\pm3$ | $-\frac{525}{65536}c(11c^2-3)s^3$ | $\frac{10395}{65536}\chi^3(11c^2\mp11c+2)s$ |
| $\pm4$ | $\frac{315}{32768}(1-11c^2)s^4$ | $-\frac{2079}{32768}\chi^4(33c^2\mp44c+13)$ |
| $\pm5$ | $-\frac{3465}{65536}cs^5$ | $-\frac{22869}{65536}\chi^4(3c\mp2)s$ |
| $\pm6$ | $\frac{1155}{262144}s^6$ | $\frac{22869}{262144}\chi^4 s^2$ |

**Table 8**. Inclination polynomials $P_{6,j,l}$ in Eq. (17) ($\chi=c\pm1$).

| $l$ | $j=3$ | $j=1$ |
|---|---|---|
| 0 | $-\frac{99099}{131072}s^6$ | $-\frac{315}{131072}(33c^4-18c^2+1)s^2$ |
| $\pm1$ | $\frac{297297}{65536}\chi s^5$ | $\frac{315}{65536}\chi(99c^4\mp66c^3-36c^2\pm18c+1)s$ |
| $\pm2$ | $-\frac{1486485}{524288}\chi^2 s^4$ | $-\frac{315}{524288}\chi^2(495c^4\mp660c^3+90c^2\pm108c-17)$ |
| $\pm3$ | $\frac{495495}{131072}\chi^3 s^3$ | $-\frac{945}{131072}\chi^2(55c^3\mp55c^2+5c\pm3)s$ |
| $\pm4$ | $-\frac{297297}{65536}\chi^4 s^2$ | $-\frac{945}{65536}\chi^2(33c^2\mp22c+1)s^2$ |
| $\pm5$ | $\frac{297297}{131072}\chi^5 s$ | $\frac{10395}{131072}(1\mp3c)\chi^2 s^3$ |
| $\pm6$ | $\frac{99099}{524288}\chi^6$ | $\frac{10395}{524288}\chi^2 s^4$ |

**Table 9**. Inclination polynomials $P_{6,j,l}$ in Eq. (17) ($\chi=c\pm1$).

| $m$ | $l$ | $S_{m,l}$ | $T_{m,l}$ |
|---|---|---|---|
| 2 | 0 | $-1+3w^2$ | 0 |
| | $\pm1$ | $-v\,w$ | $\pm uw$ |
| | $\pm2$ | $u^2-v^2$ | $\pm2u\,v$ |
| 3 | 0 | 0 | $w(5w^2-3)$ |
| | $\pm1$ | $\pm u(5w^2-1)$ | $v(5w^2-1)$ |
| | $\pm2$ | $\pm2uvw$ | $w(v^2-u^2)$ |
| | $\pm3$ | $\pm u(u^2-3v^2)$ | $-v(v^2-3u^2)$ |
| 4 | 0 | $3-30w^2+35w^4$ | 0 |
| | $\pm1$ | $vw(3-7w^2)$ | $\pm u\,w\,(-3+7w^2)$ |
| | $\pm2$ | $\frac{1}{2}(u^2-v^2)(-1+7w^2)$ | $\pm uv(-1+7w^2)$ |
| | $\pm3$ | $v(-3u^2+v^2)w$ | $\pm u(u^2-3v^2)w$ |
| | $\pm4$ | $\frac{1}{4}(u^4-6u^2v^2+v^4)$ | $\pm uv(u^2-v^2)$ |
| 5 | 0 | 0 | $w\,(15-70w^2+63w^4)$ |
| | $\pm1$ | $\pm u(1-14w^2+21w^4)$ | $v(1-14w^2+21w^4)$ |
| | $\pm2$ | $\pm2uvw(-1+3w^2)$ | $(u^2-v^2)w(1-3w^2)$ |
| | $\pm3$ | $\pm u(u^2-3v^2)(1-9w^2)$ | $v(-3u^2+v^2)(-1+9w^2)$ |
| | $\pm4$ | $\pm4uv(-u^2+v^2)w$ | $(u^4-6u^2v^2+v^4)w$ |
| | $\pm5$ | $\pm u(u^4-10u^2v^2+5v^4)$ | $v(5u^4-10u^2v^2+v^4)$ |
| 6 | 0 | $5-105w^2+315w^4-231w^6$ | 0 |
| | $\pm1$ | $v(5-30w^2+33w^4)w$ | $\mp u(5-30w^2+33w^4)w$ |
| | $\pm2$ | $(u^2-v^2)(1-18w^2+33w^4)$ | $\pm2uv(1-18w^2+33w^4)$ |
| | $\pm3$ | $v(3u^2-v^2)(3-11w^2)w$ | $\pm u(u^2-3v^2)(-3+11w^2)w$ |
| | $\pm4$ | $\frac{1}{4}(1-11w^2)(u^4-6u^2v^2+v^4)$ | $\pm(1-11w^2)uv(u^2-v^2)$ |
| | $\pm5$ | $v(5u^4-10u^2v^2+v^4)w$ | $\mp u(u^4-10u^2v^2+5v^4)w$ |
| | $\pm6$ | $(v^2-u^2)(u^4-14u^2v^2+v^4)$ | $\mp2(3u^5-10u^3v^2+3uv^4)v$ |

**Table 10**. Third-body direction polynomials in Eq. (17); $u=x^\star/r^\star$, $v=y^\star/r^\star$, $w=z^\star/r^\star$.