HYBRID SGP4: TOOLS AND METHODS

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ABSTRACT

Two-Line Elements (TLEs) continue to be the sole public source of orbiter observations. The accuracy of TLE propagations through the Simplified General Perturbations-4 (SGP4) software decreases dramatically as the propagation horizon increases, and thus the period of validity of TLEs is very limited. As a result, TLEs are gradually becoming insufficient for the growing demands of Space Situational Awareness (SSA). We propose a technique, based on the hybrid methodology, aimed at extending TLEs validity with minimal changes to the current TLE-SGP4 system. It requires the distribution of hybrid TLEs, HTLEs, which encapsulate the standard TLE and the model of its propagation error. The validity extension can be accomplished by processing HTLEs through the hybrid SGP4 propagator, HSGP4, which comprises the standard SGP4 and an error corrector.

Index Terms— Artificial satellite theory, orbit propagator, hybrid perturbation method, SGP4, TLE, SSA

1. INTRODUCTION

In the context of Space Situational Awareness (SSA), a growing catalog of thousands of objects requires fast and accurate propagations in order to compute collision probabilities. The sole universal source of observations for the SSA community is distributed as Two-Line Elements (TLEs),[1] which are generated as mean elements based on the Kozai mean motion.[1] The propagation of TLEs needs to be done through the Simplified General Perturbations-4 (SGP4) software,[2,3] which is the propagator specially adapted to TLE specifications, although it only considers the main perturbing effects. Nevertheless, no other propagator is recommended to be used with TLEs. In addition, since TLEs are based on mean elements instead of osculating elements, the associated uncertainty causes a rapid loss of accuracy, and therefore a very reduced propagation horizon. As a result, the current system based on TLEs and SGP4 is becoming insufficient for SSA growing demands.

We propose a non-intrusive enhancement of TLEs and SGP4: hybrid TLEs, HTLEs, and hybrid SGP4, HSGP4. The improvement is based on hybrid propagation, which requires the modeling of the error associated to the SGP4 propagation of a TLE during an initial control interval, and its estimation at any future instant in order to be used as a correction to the SGP4 generated ephemeris. This technique leads to a clear extension of the validity of TLEs for accurate propagations.

In order to implement this methodology, the HTLE distributor would have to model each orbiter's error by making use of osculating elements, so that some model parameters could be included in HTLEs. Accordingly, the end user would simply have to use the HSGP4 propagator, which performs a standard SGP4 propagation followed by an error correction based on the model that has been encapsulated in the HTLE.

The impact of the proposed strategy on the efficiency of the process would be very reduced. On the one hand, the information to be broadcast would be increased by just a few additional parameters. On the other hand, the computational overhead would also be small, because only a few basic operations would be necessary both in the HTLE distributor and in the end user sides.

The outline of this paper starts with a brief description of the hybrid methodology for orbit propagation in Section 2. Next, Section 3 presents the process of generating an HTLE, which requires the determination of SGP4 propagation error, Subsection 3.1, its modeling through the Holt-Winters method, Subsection 3.2, the validation and verification of the model, Subsection 3.3, and the final generation of the HTLE, Subsection 3.4. Then, the procedure to propagate an HTLE by means of HSGP4 is discussed in Section 4, which includes a brief description of the HSGP4 propagator, Subsection 4.1, and the presentation of some results, Subsection 4.2. Finally, the main concepts of the study are summarized in Section 5.

2. HYBRID METHODOLOGY

The hybrid methodology for orbit propagation is designed to complement one of the classical methods, namely general perturbation theories,[1,4–7] special perturbation theories,[8,9] or semianalytical techniques,[10–12] with time series forecasting techniques based on either statistical models[13,14] or computational intelligence methods.[15] As a result, the forecasting phase can model the dynamics missing from the integrating stage, based on accurate ephemeris available during an initial control interval, with the aim of reproducing it at future epochs, when accurate ephemeris are no longer available.

Similarly to any other propagation theory, the hybrid

[1] http://www.space-track.org/

methodology is designed to determine an estimation $\hat{\mathbf{x}}_f$ of the position and velocity of an orbiter at a final instant t_f, from its position and velocity $\mathbf{x}_1$ at an initial instant t_1, where $\mathbf{x}$ represents the complete set of six variables, which can be referred to any canonical or non-canonical frame of reference.

In the first place, an initial approximation to $\hat{\mathbf{x}}_f$ has to be calculated through the application of the integrating method $\mathcal{I}$, which represents any of the aforementioned classical theories, to the initial conditions $\mathbf{x}_1$:

$$\mathbf{x}_f^{\mathcal{I}} = \mathcal{I}(t_f, \mathbf{x}_1). \tag{1}$$

In general, the integrating method $\mathcal{I}$ includes some simplifications so as to make the process viable and affordable, and thus $\mathbf{x}_f^{\mathcal{I}}$ is just an approximation of $\hat{\mathbf{x}}_f$. A second step, in which an estimation of the difference is predicted, is therefore necessary. In order to achieve it, a forecasting technique has to model the dynamics missing from the approximation generated by $\mathcal{I}$. For that purpose, a set of either real observations or accurately generated ephemeris $\mathbf{x}$, which represent the real dynamics of the orbiter, must be available during an initial *control interval* $[t_1, t_T]$. By means of those values, the error of the integrating technique, that is, its difference with respect to the orbiter real behavior, can be determined for any instant t_i in the control interval as

$$\boldsymbol{\varepsilon}_i = \mathbf{x}_i - \mathbf{x}_i^{\mathcal{I}}. \tag{2}$$

The time series of $\boldsymbol{\varepsilon}_i$ values from t_1 to t_T, $\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_T$, which we call *control data*, contains the dynamics to be modeled and reproduced by the forecasting technique. After the adjustment process, an estimation of the error at the final instant t_f, $\hat{\boldsymbol{\varepsilon}}_f$, can be determined, thereby allowing for the calculation of the desired value of $\hat{\mathbf{x}}_f$ as

$$\hat{\mathbf{x}}_f = \mathbf{x}_f^{\mathcal{I}} + \hat{\boldsymbol{\varepsilon}}_f. \tag{3}$$

3. GENERATION OF HYBRID TLES: HTLES

In this section, the algorithm to generate a hybrid TLE, HTLE, from a standard TLE, will be described with the aid of the flow chart shown in Figure 1.

3.1. SGP4 propagation error

The first step in generating an HTLE consists in discovering the pattern of the error of the ephemeris obtained through the SGP4 propagation of the standard TLE, with respect to either precise ephemeris generated by means of an accurate numerical propagator, or real observations.

In this case, the TLE reproduced in Figure 2, which corresponds to the LEO satellite *Deimos 1*, will be processed during a period of time in absence of maneuvers so as to obtain the corresponding HTLE. The osculating orbital elements for this TLE are:

- semimajor axis: $a = 7047.253$ km,
- eccentricity: $e = 0.001229$,
- inclination: $i = 98.093°$,
- argument of perigee: $\omega = 67.201°$,
- right ascension of ascending node: $\Omega = 23.673°$,
- mean anomaly: $M = 292.869°$.

In the first place, as shown in Figure 1, the TLE is propagated with SGP4 during a control interval, from $t = 1$ to $t = T$, in order to obtain the SGP4 set of ephemeris, $\{\mathbf{x}_t^{\mathrm{SGP4}}\}_{t=1}^{T}$, where $\mathbf{x}$ represents the set of six variables.

In addition, the first obtained ephemeris, that is $\mathbf{x}_1^{\mathrm{SGP4}}$, is also propagated by means of a highly accurate orbit propagator so as to generate a set of precise ephemeris $\{\mathbf{x}_t^{\mathrm{OP}}\}_{t=1}^{T}$, which we will call *pseudo-observations*. It is worth noting that a set of real observations could also be used instead of pseudo-observations. For the present study, the numerical propagation of a model which considers a 60×60 Earth gravitational potential, atmospheric drag, luni-solar effect, solar radiation pressure including eclipses, Earth albedo, Earth IR, Earth solid tides, and relativistic effect has been used in order to generate pseudo-observations during the control interval.

Finally, a time series which represents the error of the SGP4 propagation can be obtained for each of the variables by subtracting the SGP4 set of ephemeris from the pseudo-observations:

$$\{\boldsymbol{\varepsilon}_t^{\mathbf{x}}\}_{t=1}^{T} = \{\mathbf{x}_t^{\mathrm{OP}}\}_{t=1}^{T} - \{\mathbf{x}_t^{\mathrm{SGP4}}\}_{t=1}^{T}. \tag{4}$$

Any frame of reference can be used for the variables $\mathbf{x}$; in this case we have transformed orbital elements $(a, e, i, \omega, \Omega, M)$ into Delaunay variables (l, g, h, L, G, H) through the application of the following expressions: $l = M$, $g = \omega$, $h = \Omega$, $L = \sqrt{\mu a}$, $G = \sqrt{\mu a(1 - e^2)}$, $H = \sqrt{\mu a(1 - e^2)} \cos i$.

For the present study, we have generated ephemeris with a sampling period of ten minutes, and we have taken an interval of ten satellite revolutions, which corresponds to approximately 0.7 days, as the control interval in which the SGP4 propagation error will be modeled.

3.2. Modeling through the Holt-Winters method

Figure 1 shows that it is possible to develop a model to estimate SGP4 errors for any future time, $\{\hat{\boldsymbol{\varepsilon}}_t^{\mathbf{x}}\}_{t=T+1}^{\infty}$, from the time series of the error during the control interval, $\{\boldsymbol{\varepsilon}_t^{\mathbf{x}}\}_{t=1}^{T}$, for each of the six variables. Therefore, improved ephemeris will be able to be generated for future instants by adding estimated corrections to the standard SGP4 propagation. The modeling of the time series can be done with several techniques; we have chosen the Holt-Winters method not only because of its simplicity and good results, but also due to the portability of the developed model, which constitutes a desirable quality in order to allow for the model to be integrated into HTLEs.

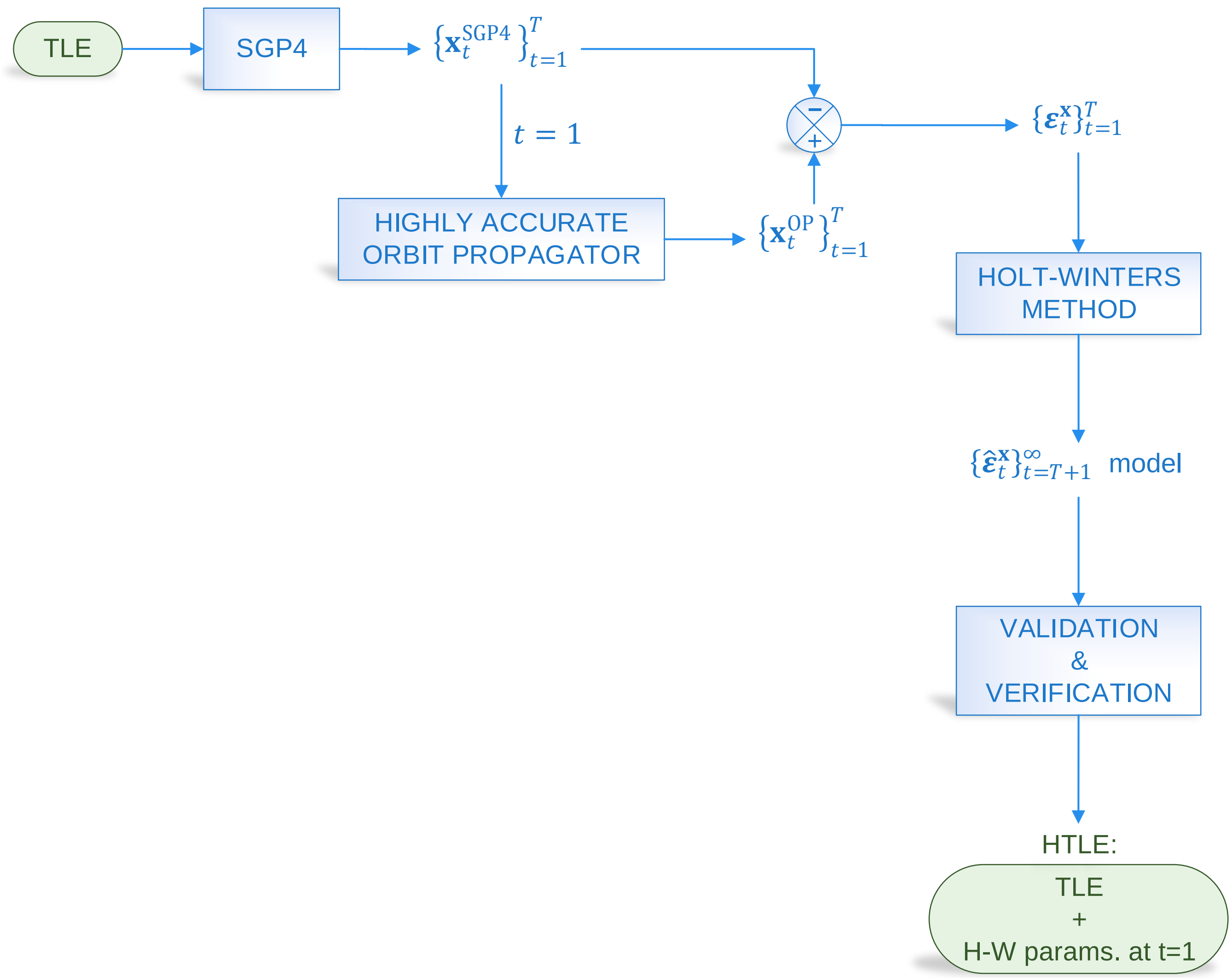

Fig. 1. Flow chart showing the process to generate a hybrid TLE, HTLE, from a standard TLE.

```
0 DEIMOS 1
1 35681U 09041A   11124.21233382  .00000325  00000-0  63164-4 0  9994
2 35681 098.0717 023.8270 0000845 081.0832 279.0474 14.69441166 94523
```

Fig. 2. Deimos 1 TLE.

Holt-Winters[16] is one of the so-called exponential smoothing methods. They consider that a time series ε_t is composed of trend μ_t, or secular variation, periodic oscillation S_t, or seasonal component, and unpredictable or random variation ν_t. In the case of an additive composition, those three components add up. In particular, the Holt-Winters method considers a linear trend $\mu_t = A + Bt$ with level A and slope B. Therefore, the value of the time series at an instant t can be estimated as

$$\hat{\varepsilon}_t = A_{t-1} + B_{t-1} + S_{t-s}, \tag{5}$$

where s represents the period of the seasonal component.

Then, A, B, and S can be determined for the same instant t as the weighted sum of two values, one based on the time series real value, ε_t, which must be known during the control interval, and the other dependent on the previous estimation of the time series components,

$$\begin{aligned} A_t &= \alpha(\varepsilon_t - S_{t-s}) + (1-\alpha)(A_{t-1} + B_{t-1}), \\ B_t &= \beta(A_t - A_{t-1}) + (1-\beta)B_{t-1}, \\ S_t &= \gamma(\varepsilon_t - A_t) + (1-\gamma)S_{t-s}. \end{aligned} \tag{6}$$

The three weights, α, β, and γ, with values in the interval $[0, 1]$, are known as *smoothing parameters*.

Algorithm 1 Holt-Winters

Require: s, c, h, and $\{\varepsilon_t\}_{t=1}^{T}$
Ensure: $\hat{\varepsilon}_{T+h|T}$
1: Estimate the values of $A_0, B_0, S_{-s+1}, \ldots, S_{-1}, S_0$
2: **for** $t = 1$; $t \leq T$; $t = t + 1$ **do**
3: $A_t = \alpha(\varepsilon_t - S_{t-s}) + (1-\alpha)(A_{t-1} + B_{t-1})$
4: $B_t = \beta(A_t - A_{t-1}) + (1-\beta)B_{t-1}$
5: $S_t = \gamma(\varepsilon_t - A_t) + (1-\gamma)S_{t-s}$
6: $\hat{\varepsilon}_t = A_{t-1} + B_{t-1} + S_{t-s}$
7: **end for**
8: Select `error_measure` $\in$ {MSE, MAE, MAPE} and express it as a function of the smoothing parameters
9: Obtain the smoothing parameters that minimize `error_measure` using the L-BFGS-B method
10: Calculate $A_T, B_T, S_{T-s+1}, \ldots, S_{T-1}, S_T$ for the optimum smoothing parameters
11: $\hat{\varepsilon}_{T+h|T} = A_T + hB_T + S_{T-s+1+h \bmod s}$
12: **return** $\hat{\varepsilon}_{T+h|T}$

The algorithm 1, which we have implemented in the R[2] statistical programming language,[17] shows how to apply the Holt-Winters method to the prediction of future time series values. The inputs to the algorithm are the amount of data per revolution, s, the number of revolutions in the control interval, c, the number of time steps after the control interval for which the time series value has to be predicted, h, and the control data, $\{\varepsilon_t\}_{t=1}^{T}$, with $T = s \times c$. The output is $\hat{\varepsilon}_{T+h|T}$, that is the forecast of the time series at the final instant $t_f = t_{T+h}$, based on the last control data, ε_T.

The algorithm starts by estimating the initial parameters A_0, B_0, $S_{-s+1}, \ldots, S_{-1}$, and S_0, which is accomplished through a classical additive decomposition into trend and seasonal variation over the first orbiter revolutions. A linear regression over the trend provides the initial level A_0 and slope B_0, whereas the seasonal component yields the values of $S_{-s+1}, \ldots, S_{-1}$, and S_0.

Then, an iterative process takes place by applying (5) and (6) to the control interval, starting from t_1 until t_T (lines 2–7). As a result, a set of time series estimated values, $\hat{\varepsilon}_t$, dependent on the smoothing parameters, are determined. Minimization of the estimation error with respect to the time series real values, ε_t, leads to the optimal values of α, β, and γ. The optimization method chosen for this process, which must allow for the imposition of constraints to the smoothing parameters, due to the fact that they must belong to the interval $[0, 1]$, is the limited memory algorithm L-BFGS-B,[18] which is a variation of the BFGS method.[19] Three different error functions can be used as an optimization criterion, mean square error, MSE, mean absolute error, MAE, and mean absolute percentage error, MAPE:

$$\begin{aligned} \text{MSE} &= \frac{1}{T}\sum_{t=1}^{T}(\varepsilon_t - \hat{\varepsilon}_t)^2, \\ \text{MAE} &= \frac{1}{T}\sum_{t=1}^{T}|\varepsilon_t - \hat{\varepsilon}_t|, \\ \text{MAPE} &= \frac{1}{T}\sum_{t=1}^{T}\left|\frac{\varepsilon_t - \hat{\varepsilon}_t}{\varepsilon_t}\right|100. \end{aligned} \tag{7}$$

Once the optimal smoothing parameters α, β, and γ have been found, the time series parameters A_T, B_T, $S_{T-s+1}, \ldots$, S_{T-1}, S_T are determined for the last period of the control data, from which the forecast time series value at the final instant, that is h epochs ahead, $\hat{\varepsilon}_f = \hat{\varepsilon}_{T+h|T}$, can be calculated as the addition of the last control data level A_T, slope B_T, and equivalent seasonal value in the previous revolution $S_{T-s+1+h \bmod s}$ (line 11).

3.3. Validation and verification

As Figure 1 shows, after generating the model of SGP4 error $\{\hat{\varepsilon}_t^{\mathbf{x}}\}_{t=T+1}^{\infty}$ for each of the six variables, a process of validation and verification takes place. Essentially, it consists in checking that the obtained models produce good results. Pseudo-observations generated after the control interval are taken as the reference in order to evaluate the distance error corresponding to the corrected ephemeris. Since not always

[2] https://www.R-project.org

is it necessary to consider all the six models to reach a good degree of accuracy, different sets of variables are checked in order to find an optimum combination both in terms of accuracy and simplicity.

In the case we are using to illustrate the process, the modeling has been performed on Delaunay variables. After checking different combinations, Table 1 compares the distance errors of the three that lead to the best results, for several time spans. It can be observed that the mean anomaly l and the argument of the perigee g, that is the terms with the faster dynamics, are the variables whose error modeling contributes the most to the accuracy of the results. Modeling also their conjugate momenta, L and G, does not have any appreciable effect, therefore they can be ignored for the sake of a simpler model. Similarly, the influence of the right ascension of the ascending node, h, and its conjugate momentum, H, is rather reduced; they can even worsen the distance error after a 30-day propagation as a consequence of being long-period terms, whose slow dynamics can scarcely be modeled in a reduced control interval of only 0.7 days. Therefore, modeling only the time series of l and g errors, ε_t^l and ε_t^g, can be considered enough to constitute the general model to be integrated into the HTLE.

3.4. Generation of the HTLE

The final step in the process is the generation of the HTLE, which consists in encapsulating the parameters deduced during the application of the Holt-Winters algorithm for each of the selected variables together with the standard TLE. These parameters can even be referred to the initial instant t_1, instead of being associated to the end of the control interval t_T, for a greater simplicity of the process. In fact, the only parameter of the model that has to be modified when moving the reference from t_T to t_1 is the level, because the slope and the seasonal component remain unchanged.

The additional parameters that the HTLE has to include, for each of the selected variables, are the level and slope of the secular component of the correction, A_1 and B_1, and the different points of the seasonal variation along one revolution, S_{-s+2}, $S_{-s+1}, \ldots, S_0$, S_1. For example, if the seasonal component is characterized with ten points, like in the case presented in this study, the number of additional parameters to be included in the corresponding HTLE is 12 per variable. As we have discussed in the previous subsection, the estimation of only two time series, $\hat{\varepsilon}_t^l$ and $\hat{\varepsilon}_t^g$, is enough in this case; therefore 24 additional parameters have to be integrated, together with the original TLE contents, into the HTLE.

4. PROPAGATION OF HTLES WITH THE HYBRID SGP4 PROPAGATOR: HSGP4

In this section we describe the steps that the hybrid SGP4 propagator, HSGP4, has to follow in order to process an HTLE, as well as the improved results that can achieve with respect to the standard SGP4 propagator.

4.1. HSGP4 propagator

Figure 3 shows a flow chart of the propagation process. As we have previously expounded, an HTLE consists of a standard TLE plus a set of Holt-Winters parameters which constitute the model of the error that SGP4 commits when propagating that TLE. Consequently, the propagation procedure to be carried out by HSGP4 must comprise two processes: a standard SGP4 propagation of the TLE, which generates the SGP4 ephemeris at the final instant t_f, $\mathbf{x}_f^{\mathrm{SGP4}}$, and a forecast of the SGP4 error at t_f based on the Holt-Winters parameters, $\hat{\boldsymbol{\varepsilon}}_f^{\mathbf{x}}$, which constitutes the correction for $\mathbf{x}_f^{\mathrm{SGP4}}$. This correction can be easily calculated by adding the secular component at t_f, determined by the straight line with level A_1 at t_1 and slope B_1, and the seasonal variation, which can be obtained through interpolation from the set of seasonal parameters S_{-s+2}, $S_{-s+1}, \ldots, S_0$, S_1. The addition of the SGP4 ephemeris and the forecast of its error leads to the improved ephemeris at t_f, $\hat{\mathbf{x}}_f$, as the output of the HSGP4 propagation process.

Evidently, both $\mathbf{x}_f^{\mathrm{SGP4}}$ and $\hat{\boldsymbol{\varepsilon}}_f^{\mathbf{x}}$ have to be added in the same frame of reference, hence some conversions may be necessary depending on the type of variables used to model SGP4 error. In our case, since we have chosen Delaunay variables for that purpose, it is necessary to convert the SGP4 ephemeris from orbital elements into Delaunay variables. Then, after adding the error forecasts $\hat{\varepsilon}_f^l$ and $\hat{\varepsilon}_f^g$, the corrected ephemeris can be transformed into any other frame of reference.

4.2. Results

Table 1, which was previously analyzed in order to evaluate the best combination of Delaunay variables to be modeled, shows not only the distance error of HSGP4 for three combinations of variables, but also the distance error of the standard SGP4 propagation with respect to the pseudo-observations generated by the numerical propagator.

The most remarkable conclusion that can be drawn from Table 1 is that the hybrid methodology clearly contributes to extend the validity of TLEs. In fact, HSGP4 distance error is five times lower than SGP4 error after one week, and twelve times lower after 30 days of propagation. The distance error of HSGP4 after 30 days, 41.1 km, is equivalent to SGP4 error after only 2.9 days.

Figure 4 depicts the distance error for both the standard SGP4 propagator and HSGP4 with modeling of ε_t^l and ε_t^g. Similarly to Table 1, Figure 4 also shows that the distance error of HSGP4 remains one order of magnitude below the distance error of the standard SGP4.

Table 1. Distance error (km) of SGP4 versus HSGP4 modeling different sets of Delaunay variables.

Propagation span	SGP4	HSGP4$_{(l,g)}$	HSGP4$_{(l,g,L,G)}$	HSGP4$_{(l,g,h,L,G,H)}$
0.7 days	10.551	1.088	0.994	1.019
1 day	14.235	3.206	3.195	3.198
2 days	28.650	5.883	5.893	5.708
7 days	101.164	20.477	20.500	19.483
30 days	486.738	41.100	41.099	42.969

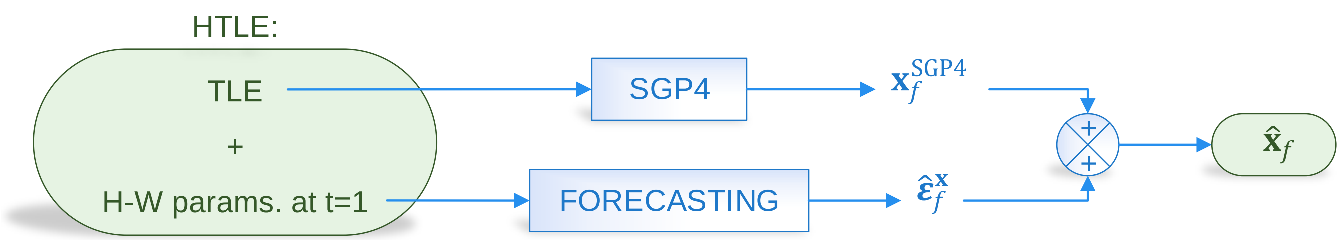

Fig. 3. Flow chart showing the procedure to process an HTLE with the hybrid SGP4 propagator, HSGP4.

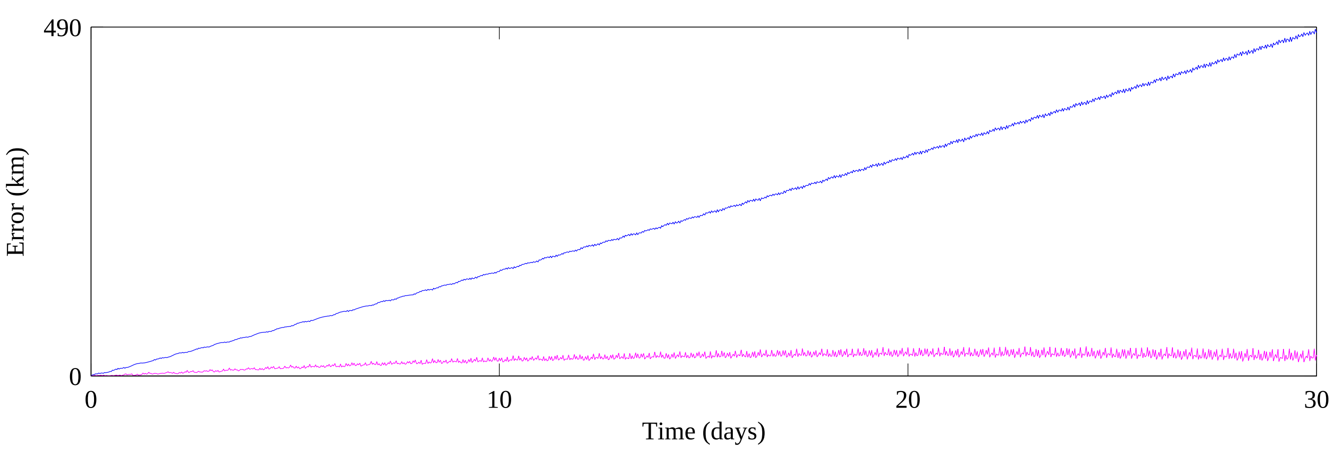

Fig. 4. Distance error corresponding to SGP4 (blue) and HSGP4 (red), modeling ε_t^l and ε_t^g, during a 30-day propagation.

5. CONCLUSION

An improvement to the well-known SGP4 propagation of TLEs has been presented, with the main aim of extending TLE validity. It is based on the application of the hybrid methodology to SGP4, which implies determining the error of the SGP4 propagation of the TLE to be processed during an initial control interval, modeling that error dynamics, and finally forecasting it for a future instant in order to correct the ephemeris generated through SGP4. The modeling process has been done according to the Holt-Winters algorithm, which allows generating a portable model that is very easy to encapsulate together with the TLE.

This approach requires the distribution of hybrid TLEs, HTLEs, which integrate both the standard TLE and the error model. Accordingly, in order to take full advantage of the HTLE extended validity, the propagation has to be done through the hybrid SGP4 propagator, HSGP4, which comprises the standard SGP4 and an error corrector that is capable of estimating the SGP4 error, based on the model encapsulated in the HTLE.

The presented methodology constitutes a non-intrusive approach, which requires minimal changes to the current SGP4-TLE system, and hence can be very easy to implement for the HTLE distributor, and to apply for the end user. In addition, despite the clear extension of the HTLE validity, the computational requirements are very low.

6. ACKNOWLEDGMENTS

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7. REFERENCES

[1] Yoshihide Kozai, "Second-order solution of artificial satellite theory without air drag," *The Astronomical Journal*, vol. 67, no. 7, pp. 446–461, September 1962.

[2] F. R. Hoots and R. L. Roehrich, "Models for propagation of the NORAD element sets," Spacetrack Report #3, U.S. Air Force Aerospace Defense Command, Colorado Springs, CO, USA, 1980.

[3] D. A. Vallado, P. Crawford, R. Hujsak, and T. S. Kelso, "Revisiting spacetrack report #3," in *Proceedings 2006 AIAA/AAS Astrodynamics Specialist Conference and Exhibit*, Keystone, CO, USA, August 2006, American Institute of Aeronautics and Astronautics, vol. 3, pp. 1984–2071, Paper AIAA 2006-6753.

[4] Dirk Brouwer, "Solution of the problem of artificial satellite theory without drag," *The Astronomical Journal*, vol. 64, no. 1274, pp. 378–397, November 1959.

[5] R. H. Lyddane, "Small eccentricities or inclinations in the Brouwer theory of the artificial satellite," *The Astronomical Journal*, vol. 68, no. 8, pp. 555–558, October 1963.

[6] Kaare Aksnes, "A second-order artificial satellite theory based on an intermediate orbit," *The Astronomical Journal*, vol. 75, no. 9, pp. 1066–1076, November 1970.

[7] Hiroshi Kinoshita, "Third-order solution of an artificial-satellite theory," Special Report no. 379, Smithsonian Astrophysical Observatory, Cambridge, MA, USA, July 1977.

[8] Hiroshi Kinoshita and Hiroshi Nakai, "Numerical integration methods in dynamical Astronomy," *Celestial Mechanics*, vol. 45, no. 1, pp. 231–244, 1989.

[9] A. C. Long, J. O. Cappellari, C. E. Velez, and A. J. Fuchs, "Goddard Trajectory Determination System (GTDS) mathematical theory (revision 1)," Tech. Rep. CSC/TR-89/6001, Computer Sciences Corporation, July 1989.

[10] J. J. F. Liu and R. L. Alford, "Semianalytic theory for a close-Earth artificial satellite," *Journal of Guidance, Control, and Dynamics*, vol. 3, no. 4, pp. 304–311, July 1980.

[11] Joseph G. Neelon, Jr., Paul J. Cefola, and Ronald J. Proulx, "Current development of the Draper Semi-analytical Satellite Theory standalone orbit propagator package," *Advances in the Astronautical Sciences*, vol. 97, pp. 2037–2052, 1998, Paper AAS 97-731.

[12] Paul J. Cefola, Donald Phillion, and Ken S. Kim, "Improving access to the semi-analytical satellite theory," *Advances in the Astronautical Sciences*, vol. 135, pp. 46 pages, 2010, Paper AAS 09-341.

[13] Juan Félix San-Juan, Montserrat San-Martín, and Iván Pérez, "An economic hybrid J_2 analytical orbit propagator program based on SARIMA models," *Mathematical Problems in Engineering*, vol. 2012, pp. 15 pages, 2012, Article ID 207381.

[14] Juan Félix San-Juan, Montserrat San-Martín, Iván Pérez, and Rosario López, "Hybrid perturbation methods based on statistical time series models," *Advances in Space Research*, vol. in press, 2015.

[15] Iván Pérez, Juan Félix San-Juan, Montserrat San-Martín, and Luis María López-Ochoa, "Application of computational intelligence in order to develop hybrid orbit propagation methods," *Mathematical Problems in*

Engineering, vol. 2013, pp. 11 pages, 2013, Article ID 631628.

[16] Peter R. Winters, "Forecasting sales by exponentially weighted moving averages," *Management Science*, vol. 6, no. 3, pp. 324–342, 1960.

[17] R Core Team, *R: A language and environment for statistical computing*, R Foundation for Statistical Computing, Vienna, Austria, 2015.

[18] Richard H. Byrd, Peihuang Lu, Jorge Nocedal, and Ciyou Zhu, "A limited memory algorithm for bound constrained optimization," *SIAM Journal on Scientific Computing*, vol. 16, no. 5, pp. 1190–1208, 1995.

[19] D. F. Shanno, "Conditioning of quasi-Newton methods for function minimization," *Mathematics of Computation*, vol. 24, no. 111, pp. 647–656, July 1970.