Asteroid Rendezvous Uncertainty Propagation

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Motivation
Rendezvous With Asteroid

Quantify the Uncertainty of a Rendezvous

- Determine Mission Success
- Seek to Quantify or Reduce Risks and Costs
- Uncertainty Quantification Can Lead to Robust Optimization
- Sensitivity of States With Respect to Inputs
State and Uncertainty Estimation

Non-linear Propagation

- Long propagation times
- Large initial uncertainty
- Tend to yield non-Gaussian posterior PDFs
- Reacquire an object
- Desire for non-intrusive approach
  - legacy software
Traditional Astrodynamic UQ

Established Techniques

- Monte Carlo
- Linearization and the state transition matrix (STM)
  - Astrodynamics Community
- Unscented Transform (UT)
  - Switching Over

These Methods Have Drawbacks

- Convergence rate of MC is slow
- STM, as well as UT, rely on Gaussian distribution assumption

Therefore, more robust methods must be considered
Proposed Astrodynaminc UQ

Methods in Development

- Polynomial Chaos Expansions (PCE)
- Gaussian Mixtures
- State Transition Tensors (STT)
- Differential Algebra (DA)

Image credit: Fujimoto, et al. (2012)

Image credit: Jones, et al. (2013)

Image credit: Horwood, et al. (2011)
Proposed Astrodynamic UQ - Properties

- PCE benefit from surrogate properties and fast convergence rate
- GMM can leverage existing filters and Gaussian techniques
- STT and DA methods reduce the computation burden

Without mitigation techniques, PCE and Gaussian Mixtures suffer from the curse of dimensionality

- Computation time increases exponentially with respect to input dimensions $d$
- Resulting in increased computation time or dimension truncation

STT Must Solve for Multiple Differential Equations, While DA is an Intrusive Method
Separated Representations
Separated Representations

**Separated Representation**

**Premise:** Decompose a multi-variate function into a linear combination of the products of uni-variate functions

\[
q(\xi_1, \ldots, \xi_d) = \sum_{l=1}^{r} s^l u_1^l(\xi_1) u_2^l(\xi_2) \cdots u_d^l(\xi_d) + O(\epsilon)
\]

- \( r \) is the separation rank
- \( u_i^l(\xi_i) \) are the unknown uni-variate functions/factors
- Computation cost dominated by relatively few MC propagations

**Extensive Background**

- Chemistry, data mining, imaging, etc
- Doostan & Iaccarino 07,09. Nouy 07,10,11,12,13.
  Doostan et al. 12,13,14. Beylkin et al. 09
Connections With SVD

Singular-value decomposition (SVD):

\[
Q = s_1^1 u_1^1 + \cdots + s_r^r u_r^r \quad \quad Q \approx \sum_{l=1}^{r} s_l^l u_1^l u_2^l
\]

Separated approx: generalization of matrix SVD to tensors:

\[
Q = s_1^1 u_1^1 + \cdots + s_r^r u_r^r \quad \quad Q \approx \sum_{l=1}^{r} s_l^l u_1^l \otimes u_2^l \otimes u_3^l
\]

Functions:

\[
q(\xi_1, \xi_2, \xi_3) = s_1^1 u_1^1 + \cdots + s_r^r u_r^r \quad \quad q \approx \sum_{l=1}^{r} s_l^l u_1^l(\xi_1) u_2^l(\xi_2) u_3^l(\xi_3)
\]
A Non-Intrusive Implementation

Problem Set Up: Given $n$ random samples

$$\xi_1, \xi_2, \ldots, \xi_N \sim \mathcal{P}_\xi(\xi)$$

Deterministic ODE Solver

reconstruct a low-rank separated representation:

$$\hat{q}(\xi) = \sum_{l=1}^{r} s^l u_1^l(\xi_1) u_2^l(\xi_2) \cdots u_d^l(\xi_d) \quad \text{s.t.} \quad \|q - \hat{q}\|_D \leq \epsilon$$

where,

$$\|\hat{q}\|_D^2 := \frac{1}{N} \sum_{j=1}^{N} \hat{q}(\xi_j)^2$$
Traditional Monte Carlo
Separated Representations

Traditional Monte Carlo
Traditional Monte Carlo
Separated Approach
Separated Approach
Separated Approach
Separated Approach
Separated Approach
Separated Approach
A Non-Intrusive Implementation

Spectral Decomposition of Factors

\[ u_i^l (\xi_i) = \sum_{p=0}^{P} c_{i,p}^l \psi_p (\xi_i) \]

where,

\[ \int_{\Gamma_i} \psi_j (\xi_i) \psi_k (\xi_i) \rho (\xi_i) d\xi_i = \delta_{jk} \]

Discrete Approximation

\[ \begin{cases} \{ c_{i,p}^l \} = \arg \min_{\{ \hat{c}_{i,p}^l \}} \left\| q (\cdot) - \sum_{l=1}^{r} s^l u_1^l (\cdot) u_2^l (\cdot) \cdots u_d^l (\cdot) \right\|_D \end{cases} \]
Separated Representations

**Computation Cost**

**Linear Scalability: Required Number of Solution Samples**

\[ \hat{q}(\xi) = \sum_{l=1}^{r} s^l \ u^l_1(\xi_1) \ u^l_2(\xi_2) \cdots u^l_d(\xi_d) \]

Number of unknowns = \( r \cdot d \cdot P \)

\[ N \sim O(r \cdot d \cdot P) \]

**Total Computation Time is Quadratic With Respect to \( d \) When ALS is Applied**

\[ C_d \sim O(K \cdot r^2 \cdot d \cdot P^2 \ (N + S)) \]

This cost should be small when compared to the number of required MC propagations.
Analysis
Distribution Characteristics

- Nominal Trajectory and Maneuver Found Using Lambert Solver
- Uncertainty in asteroid 2006 DN orbital elements and interceptor initial state
- Error in magnitude and direction of interceptor maneuver at epoch
- Propagated for 1088 days, Dormand-Prince (5)4 Integrator
- Estimate heliocentric Cartesian coordinates and velocity

Results compared to one million MC samples
### Random Inputs

<table>
<thead>
<tr>
<th>Mean</th>
<th>STD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$ (AU)</td>
<td>1.38013</td>
</tr>
<tr>
<td>$e$</td>
<td>0.27859</td>
</tr>
<tr>
<td>inc (deg)</td>
<td>0.26764</td>
</tr>
<tr>
<td>$\omega$ (deg)</td>
<td>101.24110</td>
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<tr>
<td>$\Omega$ (deg)</td>
<td>96.62356</td>
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<tr>
<td>$M$ (deg)</td>
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<table>
<thead>
<tr>
<th>Mean</th>
<th>STD</th>
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</thead>
<tbody>
<tr>
<td>$x$ (AU)</td>
<td>$-0.93808$</td>
</tr>
<tr>
<td>$y$ (AU)</td>
<td>$-0.35197$</td>
</tr>
<tr>
<td>$z$ (AU)</td>
<td>$1.9736e - 05$</td>
</tr>
<tr>
<td>$\dot{x}$ (km/s)</td>
<td>9.99105</td>
</tr>
<tr>
<td>$\dot{y}$ (km/s)</td>
<td>$-28.00263$</td>
</tr>
<tr>
<td>$\dot{z}$ (km/s)</td>
<td>2.1797e - 04</td>
</tr>
<tr>
<td>$\Delta V_x$ (km/s)</td>
<td>1.51305</td>
</tr>
<tr>
<td>$\Delta V_y$ (km/s)</td>
<td>$-3.48573$</td>
</tr>
<tr>
<td>$\Delta V_z$ (km/s)</td>
<td>$-0.05830$</td>
</tr>
<tr>
<td>$\theta$ (deg)</td>
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</tr>
<tr>
<td>$\phi$ (deg)</td>
<td>0</td>
</tr>
</tbody>
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SR Results \((d = 15)\)

15 random inputs required 1200 samples, \(r = 8\), and \(P = 3\)
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### SR Results ($d = 15$)

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<tr>
<th></th>
<th>Rel. Mean</th>
<th>Rel. STD</th>
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<tbody>
<tr>
<td>$x_{DN}$</td>
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<td>7.9e-04</td>
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<tr>
<td>$y_{DN}$</td>
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<td>1.4e-03</td>
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<tr>
<td>$z_{DN}$</td>
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<td>$\dot{x}_{DN}$</td>
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<td>2.6e-04</td>
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<td>$\dot{y}_{DN}$</td>
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<td>$\dot{z}_{DN}$</td>
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<tr>
<td>$z_{int}$</td>
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<td>4.3e-04</td>
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## Sensitivity Analysis

<table>
<thead>
<tr>
<th>Inputs</th>
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<tr>
<td>( M )</td>
<td>0.9</td>
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<tr>
<td>( M )</td>
<td>0.9</td>
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| \( x_{int} \) | \(-0\) | \(-0\) | \(-0\) | \(-0\) | \(-0\) | \(-0\) |
| \( y_{int} \) | \(-0\) | \(-0\) | \(-0\) | \(-0\) | \(-0\) | \(-0\) |
| \( z_{int} \) | \(-0\) | \(-0\) | \(-0\) | \(-0\) | \(-0\) | \(-0\) |
| \( \dot{x}_{int} \) | \(-0\) | \(-0\) | \(-0\) | \(-0\) | \(-0\) | \(-0\) |
| \( \dot{y}_{int} \) | \(-0\) | \(-0\) | \(-0\) | \(-0\) | \(-0\) | \(-0\) |
| \( \dot{z}_{int} \) | \(-0\) | \(-0\) | \(-0\) | \(-0\) | \(-0\) | \(-0\) |
| \( |\Delta V| \) | 0.8 | 0.9 | 0.9 | 0.9 | 0.8 | 0.8 |
| \( \theta \) | \(-0\) | \(-0\) | \(-0\) | \(-0\) | \(-0\) | \(-0\) |
| \( \phi \) | 0.8 | 0.8 | 0.9 | 0.8 | 0.8 | 0.9 |
Rendezvous Distance

One-to-One Comparison of One Million Samples
Rendezvous Distance

One-to-One Comparison of One Million Samples
Rendezvous Distance

One-to-One Comparison of One Million Samples

Minimum Distance Was Approximately 4400 km
Summary

Separated Representations

• Non-linear propagation of uncertainty can be expensive or complex
  - SR estimates a posterior distribution with a surrogate method
    - With a largely linear cost in $d$
  - Rendezvous PDF too sparse for target distance
    - Different approach such as all-to-all
      - Addition of TCM and optimization

Balducci, et al.
University of Colorado Boulder
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Questions and Comments
Additional Material