JOINT OPTIMIZATION OF MAIN DESIGN PARAMETERS OF ELECTRIC PROPULSION SYSTEM AND SPACECRAFT TRAJECTORY

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## Low-thrust trajectory optimization: fixed-time problems

<table>
<thead>
<tr>
<th>Problem</th>
<th>Minimum propellant</th>
<th>Minimum thrust</th>
<th>Optimal thrust&amp;power</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost function</td>
<td>$m_p \rightarrow \min \Leftrightarrow m_{PSFS} = (1+a_t)m_p \rightarrow \min$</td>
<td>$T \rightarrow \min \Leftrightarrow P \rightarrow \min \Leftrightarrow m_{PSPU} = \gamma P = \gamma Tc/2 \eta \rightarrow \min$</td>
<td>$m_u \rightarrow \max \Leftrightarrow m_{PSPU} + m_{PSFS} = \gamma P + (1+a_t)m_p \rightarrow \min$</td>
</tr>
<tr>
<td>Pros</td>
<td>- Maximum $m_u$ for fixed $T, c, P$</td>
<td>- Bounds region of solutions existance for minimum-propellant problem ($T_{\min}(c)$)</td>
<td>- Optimal $T, c, P$ are computed together with OCP solving</td>
</tr>
<tr>
<td>Cons</td>
<td>- Irregular problem: it is unknown apriori either solution exists or not</td>
<td>- Excessive propellant consumption on the minimum-thrust solution</td>
<td>- BVP has increased order (two additional optimality conditions and two additional unknowns)</td>
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</tbody>
</table>
The dynamic problem: a model of the spacecraft motion with the specified design parameters

The problem of optimizing the heliocentric trajectory of the spacecraft with EP in a central gravitational field (fixed-time problem, approach based on gravity spheres):

\[
d\mathbf{x} = \mathbf{v}
\]
\[
d\mathbf{v} = \Omega \times \mathbf{x} + \delta \frac{T}{m} \mathbf{e}_r
\]
\[
dm = -\delta \frac{T}{c}
\]

\[\Omega = \mu / r\]
\[r = |\mathbf{x}|\]
\[T, c, t_0, \Delta t \text{ - fixed}\]

Main trajectory parameters:

\[t_0, \Delta t, V_{\infty 0}, \ldots\]

A choice of controls \[\delta(t), \mathbf{e}_r(t)\] maximizing final mass of spacecraft:

\[m(t_f) \to \text{max}\]
The parametric problem: simplest model of the mass budget

Spacecraft with EPS

«Useful» part

Payload

Service systems & structure

Power supply system (PSS)

Power processing unit (PPU)

Propellant storage and feeding system (PSFS)

Thruster

Power supply & propulsion unit (PSPU)

Initial SC mass: \[ m_0 = m_u + m_{PSPU} + m_{PSFS} = m_u + \gamma P + (1+a_t)m_p \]

PSPU mass: \[ m_{PSPU} = \gamma P = \gamma Tc/2\eta \]

PSFS mass: \[ m_{PSFS} = (1+a_t)m_p \]

Main design parameters: \[ m_0, \eta, \gamma, a_t \text{ (fixed)}, \]
\[ P, T, c: P = Tc/2\eta \text{ (selected)} \]
General formulation of thrust minimization problem

The region of existence of a solutions is bounded by the values of minimum thrust and exhaust velocity. A certain finite increment in characteristic velocity is required in order to complete the intended flight. This increment may be attained at fixed time $\Delta t = t_f - t_0$ only by applying a sufficient thrust.

Since the allowed propellant consumption is often limited from above:

$$m_p \leq m_{p\text{max}} < m_0, \quad \text{or} \quad m(t_f) \geq m_{f\text{min}} < m_0$$

consequently, exhaust velocity should be sufficiently high.

Compute optimal $\delta(t), e_T(t), T, c$, minimizing useful mass of spacecraft

$$J = T \rightarrow \text{min},$$

$$\frac{dx}{dt} = v,$$

$$\frac{dv}{dt} = \Omega_x + \delta \frac{T}{m} e_T,$$

$$\frac{dm}{dt} = -\delta \frac{T}{c}$$
Boundary value problem of PMP

Smoothed thrustig function

\[ \delta = \frac{1}{2} \left( \frac{\psi}{|\psi| + \varepsilon} + 1 \right) \]

\( \varepsilon \) - the small regularizing parameter

Differential equations of optimal motion

\[ \frac{dx}{dt} = v \]
\[ \frac{dv}{dt} = \Omega_x + \delta \frac{T}{m} e_T \]
\[ \frac{dm}{dt} = -\delta \frac{T}{c} \]
\[ \frac{dT}{dt} = 0 \]

Initial conditions

\[ x(t_0) = x_0(t_0) \]
\[ v(t_0) = v_0(t_0) + V_{\infty} \frac{p_v(t_0)}{p_v(t_0)} \]
\[ m(t_0) = m_0 \]
\[ p_T(t_0) = 0 \]

Final conditions

\[ x(t_f) = x_f(t_f) \]
\[ v(t_f) = v_f(t_f) \]

Unknown BVP parameters

\[ (p_x(t_0), p_v(t_0), p_m(t_0), T) \]

Necessary conditions of thrust minimality

\[ p_T(t_f) = \text{const} < 0 \]
Minimum of PSPU+PSS mass wrt. specific impulse and thrust

\[ m_{\text{PSPU}} = \gamma Tc/2 \eta \]

\[ m_{\text{PSFS}} = (1+a_t)m_p = (1+a_t)e^{(-V_{id}/c)} \]
General formulation of joint optimization problem

Compute optimal $\delta(t), e_T(t), T, c$, maximizing useful mass of spacecraft

$$m_u = m_0 - \gamma P - (1 + a_t)m_p$$

where $P = \frac{Tc}{2\eta}$ - electrical power of EPS

$$m_p = m_0 - m(t_f) = \int_{t_0}^{t_f} \delta \frac{T}{c} dt$$ - active propellant mass

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = \Omega_x + \delta \frac{T}{m} e_T$$

$$\frac{dm}{dt} = -\delta \frac{T}{c}$$

$t = t_0 :$

$x(t_0) = x_0(t_0)$

$|v(t_0) - v_0(t_0)| = V_{\infty 0}$

$m(t_0) = m_0$

$t = t_f = t_0 + \Delta t :$

$x(t_f) = x_f(t_f)$

$v(t_f) = v_f(t_f)$
Dependency of specific impulse on power and thrust

\[ c = \frac{2\eta P}{T} \]

Equations of motion

\[
\begin{align*}
\frac{dx}{dt} &= v \\
\frac{dv}{dt} &= \Omega_x + \delta \frac{T}{m} e_T \\
\frac{dm}{dt} &= -\delta \frac{T^2}{2\eta P} \\
\frac{dT}{dt} &= 0 \\
\frac{dP}{dt} &= 0
\end{align*}
\]

Cost function

\[
J = \int_{t_0}^{t_f} \left( \frac{\gamma P}{\Delta t \cdot (1 + a_i)} + \delta \frac{T^2}{2\eta P} \right) dt = \int_{t_0}^{t_f} \left( \alpha P + \delta \frac{T^2}{2\eta P} \right) dt
\]

Hamiltonian

\[
H = -\alpha P + p_x^T v + p_v^T \Omega_x + \delta T \psi
\]

Optimal control

\[
e_T = \frac{p_v}{p_v} \\
\delta = \begin{cases} 
1, \psi > 0 \\
0, \psi < 0 
\end{cases} \\
\psi = \frac{p_v}{m} - \frac{T \cdot (p_m + 1)}{2\eta P}
\]
Boundary value problem of PMP

Smoothed thrusting function
\[
\delta = \frac{1}{2} \left( \frac{\psi}{|\psi| + \epsilon(\tau)} + 1 \right)
\]

Differential equations of optimal motion
\[
\begin{align*}
\frac{dx}{dt} &= v \\
\frac{dv}{dt} &= \Omega_x + \delta \frac{T}{m} p_v \\
\frac{dm}{dt} &= -\delta \frac{T^2}{2\eta P} \\
\frac{dT}{dt} &= 0 \\
\frac{dP}{dt} &= 0
\end{align*}
\]

Initial conditions
\[
t = t_0 : \\
x(t_0) = x_0(t_0) \\
v(t_0) = v_0(t_0) + V_{\infty} \frac{p_v(t_0)}{p_v(t_0)} \\
m(t_0) = m_0 \\
p_T(t_0) = 0 \\
p_P(t_0) = 0
\]

Final conditions
\[
t = t_f = t_0 + \Delta t : \\
x(t_f) = x_f(t_f) \\
v(t_f) = v_f(t_f) \\
p_m(t_f) = 0
\]

Unknown BVP parameters
\[
(p_x(t_0), p_v(t_0), p_m(t_0), T, P)
\]

Necessary conditions of thrust and power optimality
\[
\begin{align*}
p_T(t_f) &= 0 \\
p_P(t_f) &= 0
\end{align*}
\]
BVP solving

Additional optimality conditions:

\[
\int_{t_0}^{t_f} \left( -\delta \psi + \delta T \frac{p_m + 1}{2\eta P} \right) = 0, \\
\int_{t_0}^{t_f} \left( \alpha - \delta T^2 \frac{p_m + 1}{2\eta P^2} \right) = 0. \\
\alpha = \frac{\gamma}{\Delta t \cdot (1 + a_t)}
\]

The method of continuation on parameter (Newton homotopy) is used to solve the boundary value problem by reducing it to a Cauchy problem.

The solution of the power-limited trajectory optimization problem is used as the initial approximation for the unknown parameters of the boundary value problem.

The initial value of the thrust is assumed to be the average value of the thrust on the solution to the power-limited problem.
Numerical example: trajectory to Mars (1/5)

Region of existence of a solution and the optimal values of the specific impulse and the thrust of the electric propulsion system (markers) for different values of the design parameters characterizing the level of the technology being used.
Numerical example: trajectory to Mars (2/5)

Isolines of the useful mass
### Numerical example: trajectory to Mars (3/5)

#### Trajectories with $V_\infty = 0$

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$\gamma$</th>
<th>$a_t$</th>
<th>$c$</th>
<th>$T$</th>
<th>$m(t_f)$</th>
<th>$m_p$</th>
<th>$m_{PSPU}$</th>
<th>$m_u$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>40</td>
<td>0.1</td>
<td>2869</td>
<td>222.66</td>
<td>809.95</td>
<td>190.05</td>
<td>388.04</td>
<td>611.96</td>
<td>4474</td>
</tr>
<tr>
<td>0.5</td>
<td>40</td>
<td>0.1</td>
<td>2374</td>
<td>220.39</td>
<td>775.11</td>
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<td>0.5</td>
<td>80</td>
<td>0.1</td>
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<td>681.77</td>
<td>318.23</td>
<td>615.33</td>
<td>384.67</td>
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<tr>
<td>0.5</td>
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<td>0.2</td>
<td>2494</td>
<td>221.02</td>
<td>784.69</td>
<td>215.31</td>
<td>474.61</td>
<td>525.39</td>
<td>5406</td>
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</tbody>
</table>

#### $V_\infty$ impact

<table>
<thead>
<tr>
<th>$V_\infty$</th>
<th>$T$</th>
<th>$c$</th>
<th>$m(t_f)$</th>
<th>$m_p$</th>
<th>$m_{PSPU}$</th>
<th>$m_u$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>m/s</td>
<td>mN</td>
<td>s</td>
<td>kg</td>
<td>kg</td>
<td>kg</td>
<td>kg</td>
<td>W</td>
</tr>
<tr>
<td>0</td>
<td>222.66</td>
<td>2869</td>
<td>809.95</td>
<td>190.05</td>
<td>388.04</td>
<td>611.96</td>
<td>4474</td>
</tr>
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<td>1000</td>
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<td>3206</td>
<td>849.56</td>
<td>150.44</td>
<td>309.92</td>
<td>690.08</td>
<td>3611</td>
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<tr>
<td>2000</td>
<td>137.08</td>
<td>3214</td>
<td>869.71</td>
<td>130.29</td>
<td>266.75</td>
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<td>3000</td>
<td>137.43</td>
<td>2997</td>
<td>879.46</td>
<td>120.54</td>
<td>247.99</td>
<td>752.01</td>
<td>2885</td>
</tr>
</tbody>
</table>
Numerical example: trajectory to Mars (4/5)

The dependence of mass from the flight duration at optimum values of thrust and specific impulse

The dependence of optimum values of thrust and specific impulse from the flight duration
Numerical example: trajectory to Mars (5/5)

Departure date – 20.04.2035

\[ m_0 = 1000 \text{ kg} \]
\[ \eta = 0.5 \]
\[ \gamma = 40 \text{ kg/kW} \]
\[ a_t = 0.1 \]
\[ \Delta t_{opt} = 382.53 \text{ days} \]
\[ c_{opt} = 2940 \text{ sec} \]
\[ T_{opt} = 216.75 \text{ mN} \]
\[ P_{opt} = 4465 \text{ W} \]
\[ m_p = 187.6 \text{ kg} \]
\[ m_u = 615.04 \text{ kg} \]
Conclusion

- It is demonstrated that the refusal of the separation on the dynamic and parametric parts of the optimization problem can significantly reduce the computation consumption.
- Necessary conditions for the optimality are received.
- The numerical method for the thrust minimization problem and the joint optimization are developed.