A Series for the Collision Probability in the Short-Encounter Model

R. García-Pelayo & J. Hernando-Ayuso

Universidad Politécnica de Madrid & The University of Tokyo

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- **This presentation is available at https://sites.google.com/site/ricardogarciapelayo/**

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- **The numbering of the equations in this presentation is the one of the article of the same title at the Journal of Guidance, Dynamics and Control, 2016.**

Summary

The probability of collision between two orbiting objects is needed to decide if an evasive maneuver is to be performed, provided there is control of at least one of them.
A series expansion for the computation of the probability of collision between two objects of known probability density function (pdf) of the position is provided. These pdf's do not need to be Gaussian. When these pdf's are Gaussian this series expansion yields a very fast computation.

In the short encounter model the collision takes place in a plane (the encounter plane) perpendicular to the relative velocity of the two objects which may collide. The objects are substituted by the smallest spheres (the "enveloping spheres", of radii R_1 and R_2) which contain them. This is a conservative approximation for the probability of collision.
Collision takes place whenever the centers of the enveloping spheres are closer than $R_1 + R_2$. Therefore

$$p = \int d\mathbf{r_1}\ \rho_1(\mathbf{r_1}) \int_{|\mathbf{r_2}-\mathbf{r_1}|<R_1+R_2} d\mathbf{r_2}\ \rho_2(\mathbf{r_2}). \tag{4}$$

General case

Taylor expansion and spherical symmetry leads to

$$p = \sum_{i=0}^{\infty} \left(\frac{R_1 + R_2}{2} \right)^{2i+2} \frac{4\pi}{(i+1)!\ i!} \int d\mathbf{r_1}\ \rho_1(\mathbf{r_1}) \sum_{j=0}^{i} \binom{i}{j} \frac{\partial^{2i} \rho_2(\mathbf{r_1})}{\partial x^{2(i-j)} \partial y^{2j}}. \tag{9}$$

x and y are Cartesian coordinates in the encounter plane.

General case

Equation (4) has a symmetry, which is that it only depends on the distribution of the difference in the positions of the two objects. It is convenient, both analytically and conceptually, to pretend that one of the positions is known exactly and transfer all the incertitude to the other position. This leads to the simplification

$$p = \sum_{i=0}^{\infty} \left(\frac{R_1 + R_2}{2} \right)^{2i+2} \frac{4\pi}{(i+1)(i!)^2} \sum_{j=0}^{i} \binom{i}{j} \frac{\partial^{2i} \rho(\langle \mathbf{r} \rangle_1 - \langle \mathbf{r} \rangle_2)}{\partial x^{2(i-j)} \partial y^{2j}}, \tag{15}$$

where $\rho \equiv (\rho_1 \circ (-1)) \otimes \rho_2$, $\otimes$ being the convolution product, defined by $(f_1 \otimes f_2)(\mathbf{r}) \equiv \int \mathrm{d}^3 r' \; f_1(\mathbf{r} - \mathbf{r}') f_2(\mathbf{r}')$, and the angular brackets denote expected value.

Gaussian case

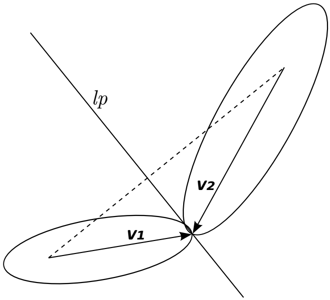
When both ρ_1 and ρ_2 are Gaussians,

$$G(\mathbf{r}_0) \equiv (\rho_1 \circ (-1)) \otimes \rho_2)(\mathbf{r_0}) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{-\frac{1}{2}\left(\frac{x_0^2}{\sigma_x^2} + \frac{y_0^2}{\sigma_y^2}\right)\right\}, \tag{17}$$

where $\mathbf{r_0}$ is the expected relative position of the objects: $\mathbf{r_0} \equiv \langle\mathbf{r}\rangle_1 - \langle\mathbf{r}\rangle_2$. In this case the probability of collision simplifies to

$$\begin{aligned} p = \sum_{i=0}^{\infty} \left(\frac{R_1 + R_2}{2}\right)^{2i+2} \frac{4\pi}{(i+1)(i!)^2} \\ \sum_{j=0}^{i} \binom{i}{j} \frac{H_{2(i-j)}(x_0/\sigma_x)\ H_{2j}(y_0/\sigma_y)}{\sigma_x^{2(i-j)}\sigma_y^{2j}}\ G(\mathbf{r}_0), \end{aligned} \tag{18}$$

where H_n is the n-th probabilists' Hermite polynomial.

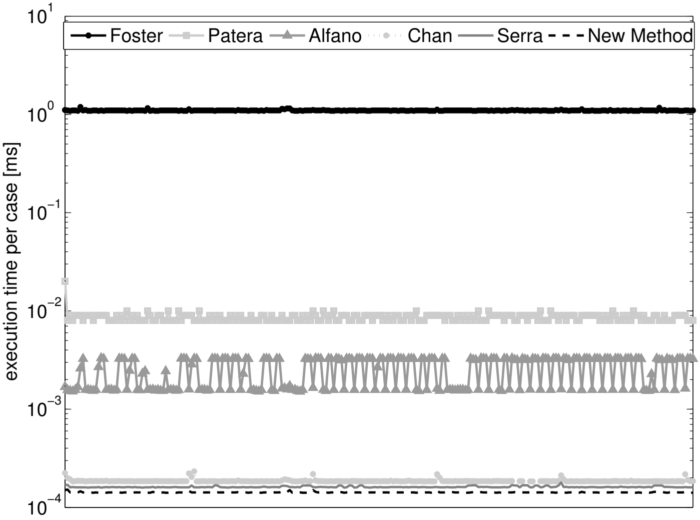


The error ellipse of the relative positions has one semiaxis which ranges, in units of the combined radius $R_1 + R_2$, from 4 to 4096 (*lp*) and another semiaxis which ranges from 4 to 256 (*sp*). The threshold values of the collision probability to make an evasive maneuver are usually between 10^{-3} and 10^{-5}. We have explored the range $10^{-7} - 10^{-1}$. All together, we have sampled 244 cases.

We do a numerical study of the convergence of the series for the Gaussian case. In all the 244 considered cases, the first two terms of the series (18) were necessary and sufficient to guarantee a 10% tolerance, that is

$$p \approx \frac{2}{\sigma_x \sigma_y} \exp\left\{-\frac{1}{2}\left(\frac{x_0^2}{\sigma_x^2} + \frac{y_0^2}{\sigma_y^2}\right)\right\}\left(\frac{R_1 + R_2}{2}\right)^2 \left(1 + \frac{1}{2}\left(\frac{\left(\frac{x_0}{\sigma_x}\right)^2 - 1}{\sigma_x^2} + \frac{\left(\frac{y_0}{\sigma_y}\right)^2 - 1}{\sigma_y^2}\right)\left(\frac{R_1 + R_2}{2}\right)^2\right). \tag{19}$$

The second term is also a bound for the absolute error.



Foster
Patera
Alfano
Chan
Serra
New Method
execution time per case [ms]
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}