A fast an efficient algorithm for the computation of distant retrograde orbits

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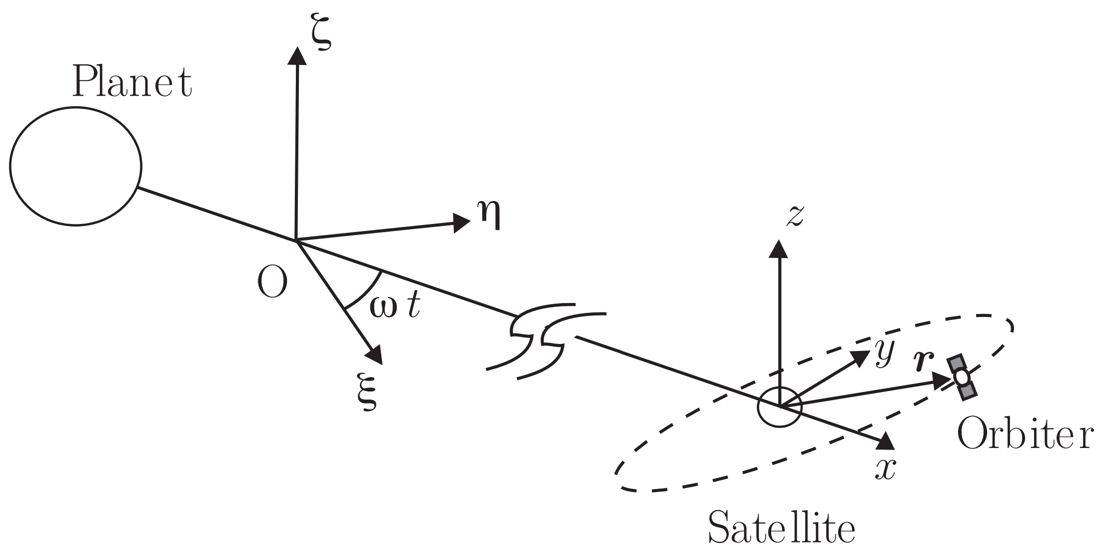
Background: Distant Retrograde Orbits

- DRO: co-orbital motion relative to the heavier primary
  - same semi-major axis but slightly different eccentricity
  - orbit about lighter primary out of its sphere of influence
  - strong stability characteristics: quarantine orbits
  - science orbit for objects with very low mass
- DePhine proposal to ESA's Cosmic Vision Program (Oberst et al. 2017, EPSC2017–539)
- NASA's Asteroid Redirect Mission (Abell et al., Lunar & Planetary Sci. Conference, 2017)
- Numerical computation of (quasi) PO's of the RTBP
- Analytical sol.: rough approximations (qualitative dynamics)
  - improvements by perturbation methods

Outline

- Recall basic facts of the Hill problem
  - useful to test feasibility of the perturbations approach
- Perturbation arrangement: distant retrograde orbits
  - perturbed linear dynamics
- Low order analytical solution
  - provides orbit design parameters
- Higher order analytical solution
  - Lindstedt series: captures the case of large librations
- Sample applications
- Conclusions

The Hill problem



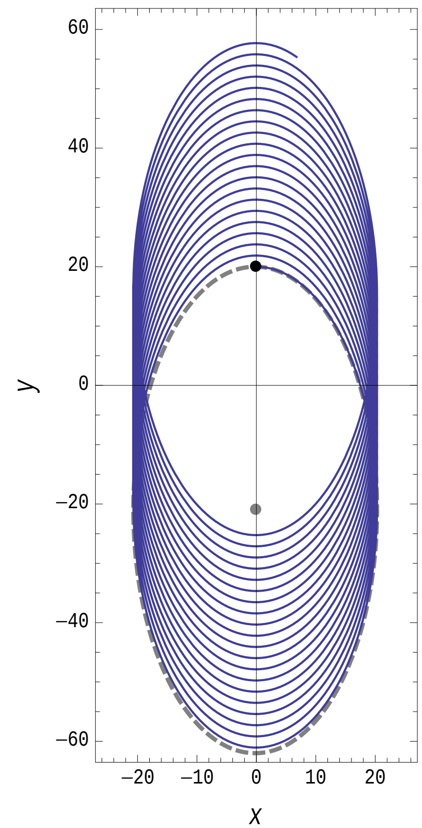
- CR3BP: primaries M, $m < M$, S/C of negligible mass
  - relative motion: rotating frame centered in the satellite
  - simplifications: $m \ll M$, $r \ll d$, d planet-satellite distance
- Hamiltonian formulation $\mathcal{H} = \mathcal{H}(\boldsymbol{X}, \boldsymbol{x})$
  - conjugate momenta $\boldsymbol{X} = \dot{\boldsymbol{x}} + \boldsymbol{\omega} \times \boldsymbol{x}$

DRO: Perturbation of the linear motion

- $\mathcal{H} = \frac{1}{2}(X^2 + Y^2 + Z^2) - \omega(xY - yX) - \frac{\omega^2}{2}(3x^2 - r^2) - \frac{\mu}{r}$
  - quadratic part (linear motion) integrable
    - relative motion of two bodies orbiting the third one with Keplerian motion
    - Clohessy-Wiltshire equations (J Aerospace Sci 1960)
  - perturbation: Keplerian potential $-\frac{\mu}{r}$ (or $-\frac{1}{r}$ in Hill units)
- Perturbation: only if r is large enough in Hill units
  - Hill units $r > 1 > 3^{-1/3} \approx 0.7$
  - motion out of Hill's sphere $\Rightarrow$ *co-orbital motion*
- Trouble if r is not too large (close encounters)

- Focus on the planar case
  - unperturbed motion: drifting ellipse
- Change to epicyclic variables:
  $(x, y, X, Y) \longrightarrow (\phi, q, \Phi, Q; \omega)$
  - semi-axis $a = 2b$ (y axis)
  - $b = \sqrt{2\frac{\Phi}{\omega}}$
  - eccentricity $k = \sqrt{\frac{3}{4}}$
  - ϕ: phase of the ellipse

    (Benest, Celest Mech 13, 1976)
- Guiding center
  - $x_C = Q/(k\omega)$ constant
  - $y_C = 2kq = 2k(q_0 - Qt)$ linear motion
  - iicc such that $Q = 0 \Rightarrow$ periodic motion



- Perturbed motion: drift may change to oscillatory motion
  - *epicycle*: S/C travels along a large ellipse
  - *deferent*: center of the epicycle moves on a small ellipse
- Approximate solution by perturbation methods
- $\mathcal{H} = \mathcal{H}(\phi, q, \Phi, Q) = \omega\Phi - (1/2)Q^2 - \mu/r, \qquad r \equiv r(\phi, q, \Phi, Q)$
  - $\mu \to 0$ (or r very large) linear growing of ϕ and q
  - drifting ellipse in the y axis direction (Clohessy-Wiltshire)
- Remove short-period effects by averaging ϕ:
  $(\phi, q, \Phi, Q) \xrightarrow{\mathcal{T}} (\phi', q', \Phi', Q'; \epsilon)$ / $\mathcal{H} \circ \mathcal{T} = \mathcal{H}(-, q', \Phi', Q') + \mathcal{O}(\epsilon^n)$
  - $\mathcal{T}$ depends on special functions
- Evolution equations (in *mean* elements) are integrable
  - low order of ϵ: harmonic oscillator
  - improved solution: Duffing oscillator (elliptic functions)
  - high order of ϵ: solution by Lindstedt series

Low order solution

- $\mathcal{K}' = \omega\Phi' - \frac{1}{2}[Q'^2 + \Omega^2 q'^2] + \mathcal{P}(-, q', \Phi', Q') + \mathcal{O}(\epsilon^n)$
  - libration frequency $\Omega \equiv \Omega(\Phi'; \mu, \omega)$
  - Φ' constant, decoupled dynamics
- Neglect $\mathcal{P}$: On average
  - (q', Q'): harmonic motion of frequency Ω

    $q' = q'_0 \cos\Omega t - (Q'_0/\Omega)\sin\Omega t,$

    $Q' = Q'_0 \cos\Omega t + \Omega q'_0 \sin\Omega t,$
  - ϕ' linear growth modulated with long-period oscillations
    * standard quadrature $\phi' = \phi'_0 + \tilde{\omega} t + \frac{\Omega}{\omega}[p(t) - p(0)]$

      $\tilde{\omega} = \omega[1 + (\Omega^2/\omega^2)d], \qquad d \equiv d(q'_0, Q'_0, \Phi') > 0$

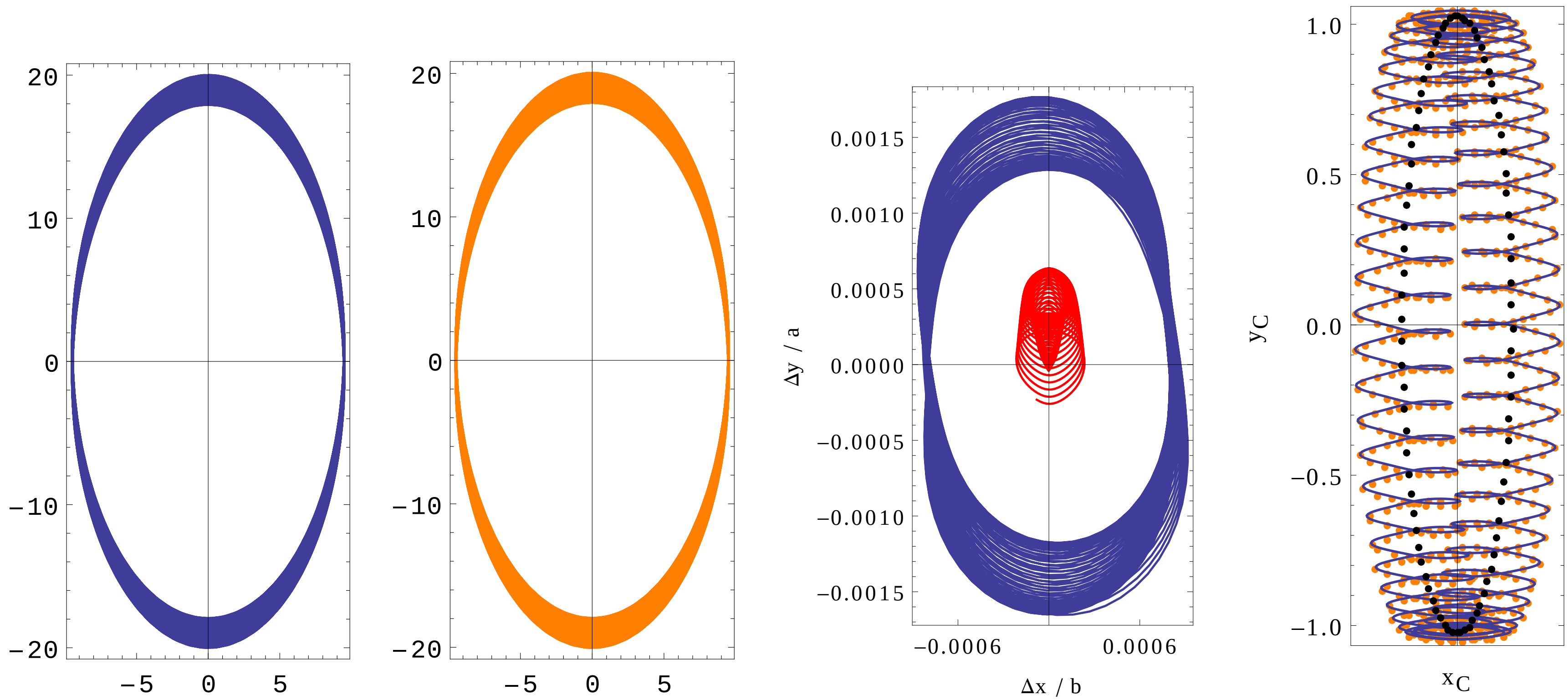
      $p = \frac{q'_0(Q'_0/\Omega)}{(b/k)^2}\cos 2\Omega t + \frac{{q'_0}^2 - (Q'_0/\Omega)^2}{2(b/k)^2}\sin 2\Omega t,$
    * $k = \sqrt{3/4}$ eccentricity of the epicycle

- Deferent evolves with harmonic oscillations

  $x_C = \frac{\Omega}{k\omega} M \sin(\Omega t + \psi), \quad y_C = 2kM \cos(\Omega t + \psi),$

  - $M = \sqrt{{q'_0}^2 + (Q'_0/\Omega)^2}, \quad \tan\psi = Q'_0/(\Omega q'_0)$
  - $x_C = 0 \Rightarrow \max(y_C) = 2kM$
- Orbit design parameters
  - $a = 2b$: size of the epicycle
  - min. y distance to the origin for $x = 0$: $d_y = a - 2kM$
  - compute iicc of $\max(y_C)$: $2k\sqrt{{q'_0}^2 + (Q'_0/\Omega)^2} = a - d_y$
- Periodic orbit (on average): commensurability between
  - orbital period $T_O = 2\pi/\tilde{\omega}$,
  - libration period $T_L = 2\pi/\Omega$
- Osculating: improve periodicity by differential corrections

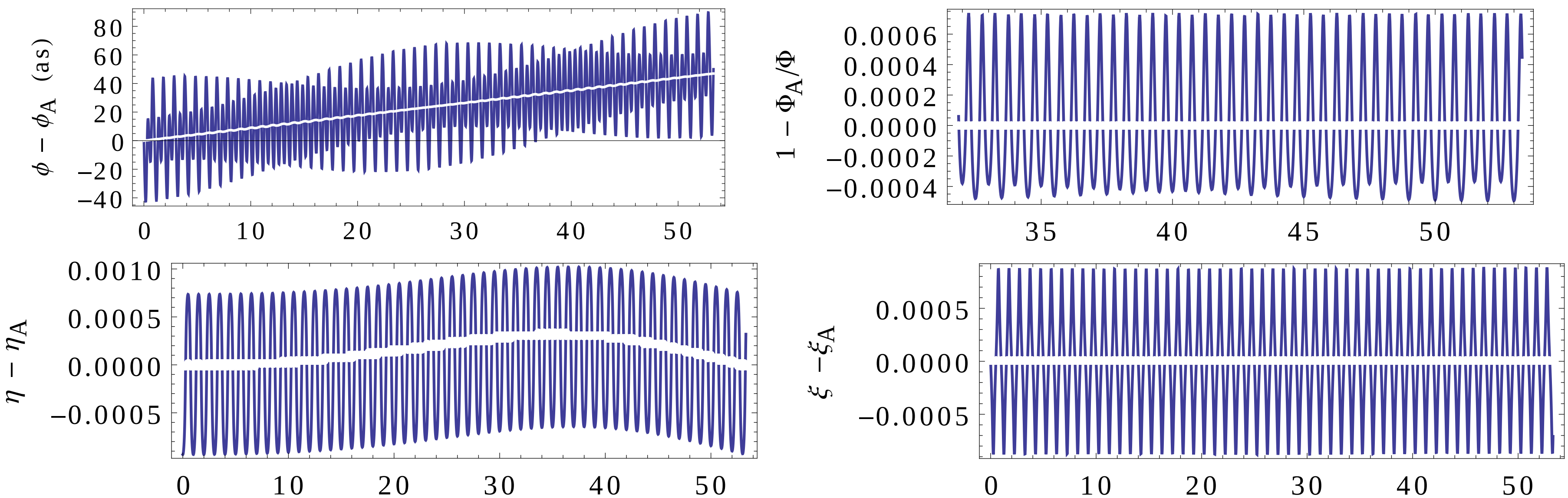
- Low order solution very good for small librations
  - significance of short-period terms



  - initial conditions $(0.1, 20, -10.5, -0.1)$

- Errors: epicyclic variables

  Blue: mean elements. White: osculating elements



High order solution

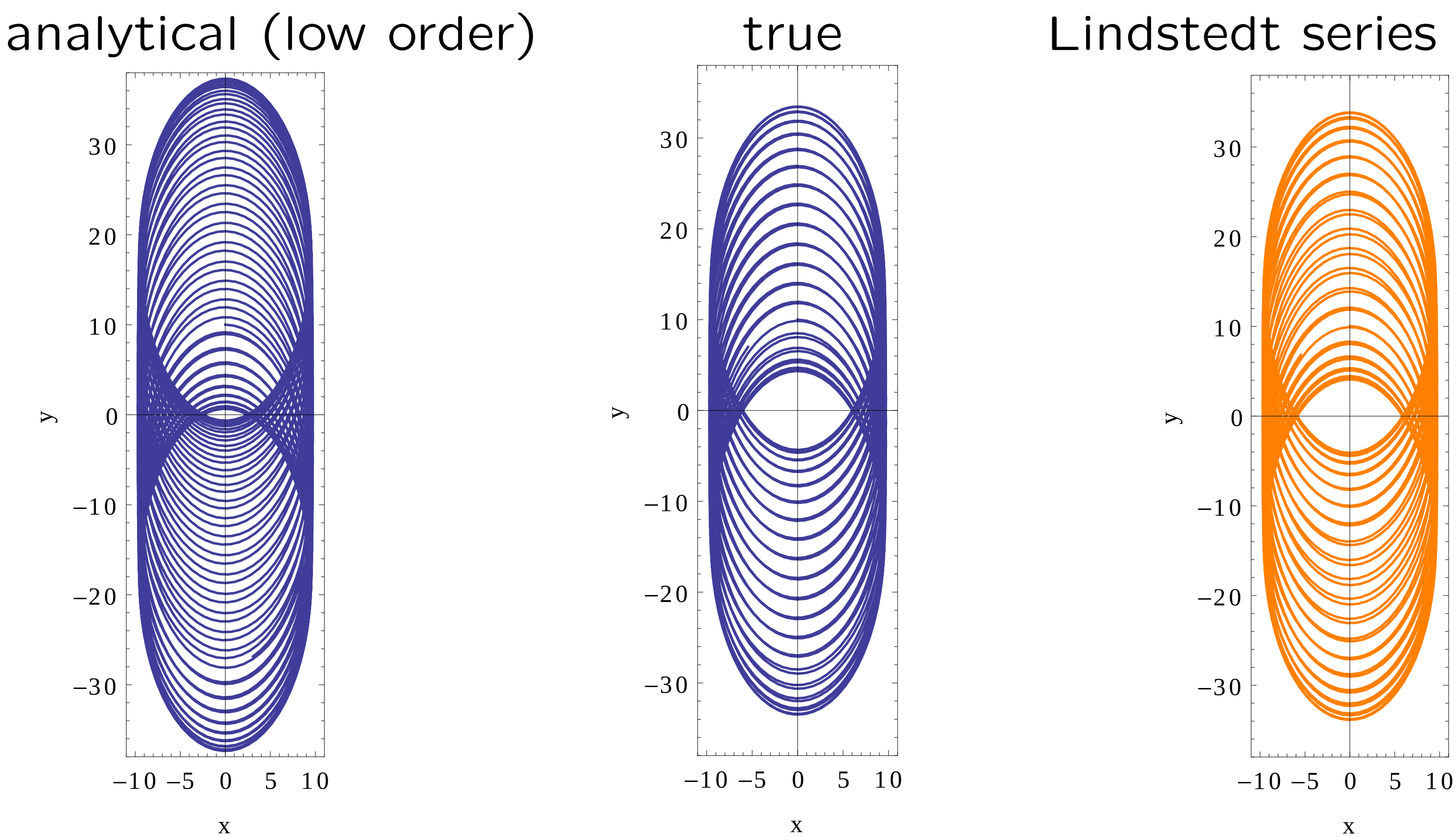
- Low order: great insight / not so good for large librations
  - poor prediction of the amplitude and libration period
- Higher orders of the perturbation approach: Lindstedt series
  - change ind. variable $\tau = nt$, replace $n = \sum_{i \geq 0} \epsilon^i n_i$
  - replace $q = \sum_{i \geq 0} \epsilon^i q_i(\tau)$, $\quad Q = \sum_{i \geq 0} \epsilon^i Q_i(\tau)$
  - chain of differential systems solved sequentially
- 5 series required:
  - time scale in which the Lindstedt series evolve (1 series)
  - time evolution of the guiding center q', Q' (2 series)
  - phase ϕ': linear growing + modulation (2 series)
    - linear freq. $\tilde{\omega}$ with which the S/C evolves (1 series)
    - long-period modulation of the S/C's phase ϕ' (1 series)
- Improved orbital and libration periods

- Lindstedt series: arranged in the form of Fourier series in Ω

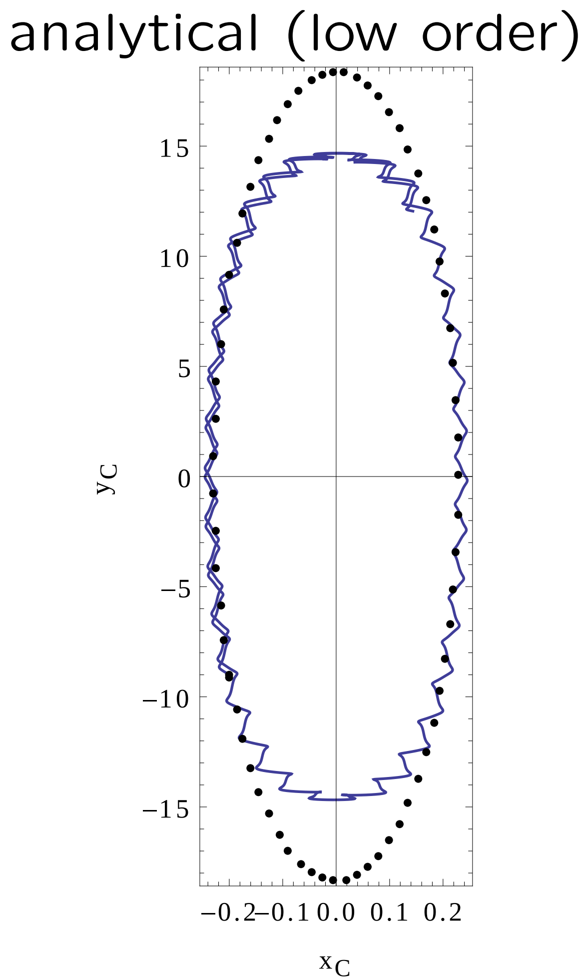
$$
\begin{aligned}
n &= \sum_{m=0}^{2}\sum_{j=0}^{m}\sum_{k=0}^{m-j}\left(\frac{\Omega}{\omega}\right)^{2(m-j-k)}\left(\frac{Q_0'/\Omega}{b}\right)^{2j}\left(\frac{q_0'}{b}\right)^{2k} n_{m,j,k} \\
q' &= \sum_{m=0}^{2}\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{m-j}\left(\frac{\Omega}{\omega}\right)^{2(m-j-k)}\left(\frac{Q_0'/\Omega}{b}\right)^{2j}\left(\frac{q_0'}{b}\right)^{2k} \\
&\quad \times\left[c_{m,i,j,k} q_0' \cos(2i+1)\Omega\tau + s_{m,i,j,k}(Q_0'/\Omega)\sin(2i+1)\Omega\tau\right] \\
Q' &= \sum_{m=0}^{2}\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{m-j}\left(\frac{\Omega}{\omega}\right)^{2(m-j-k)}\left(\frac{Q_0'/\Omega}{b}\right)^{2j}\left(\frac{q_0'}{b}\right)^{2k} \\
&\quad \times\left[C_{m,i,j,k} Q_0' \cos(2i+1)\Omega\tau + S_{m,i,j,k}(q_0'\Omega)\sin(2i+1)\Omega\tau\right]
\end{aligned}
$$

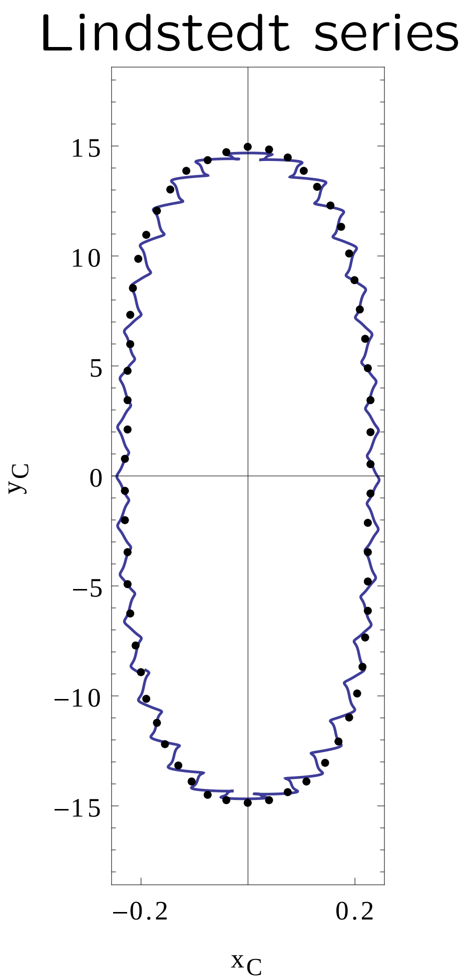
- Additional series for the phase ϕ'
- Quite effective evaluation

Sample orbit with large libration: $(0, 10, -0.5, 1)$



- Guiding center: $(0, 10, -0.5, 1)$

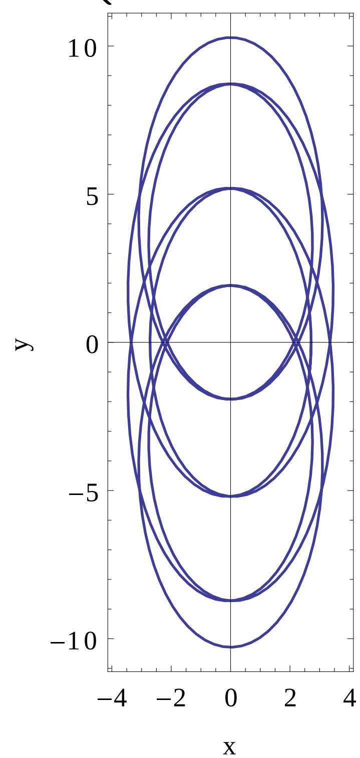




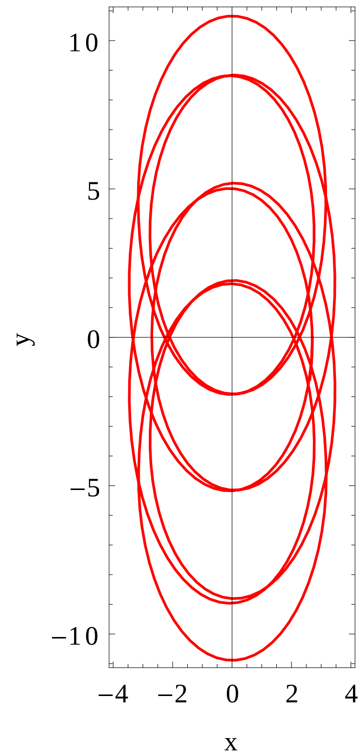
Close approach to the origin

- Periodic orbit of initial conditions $(2.7163, 0, 0, -2.9724)$

true (numerical)



Lindstedt series

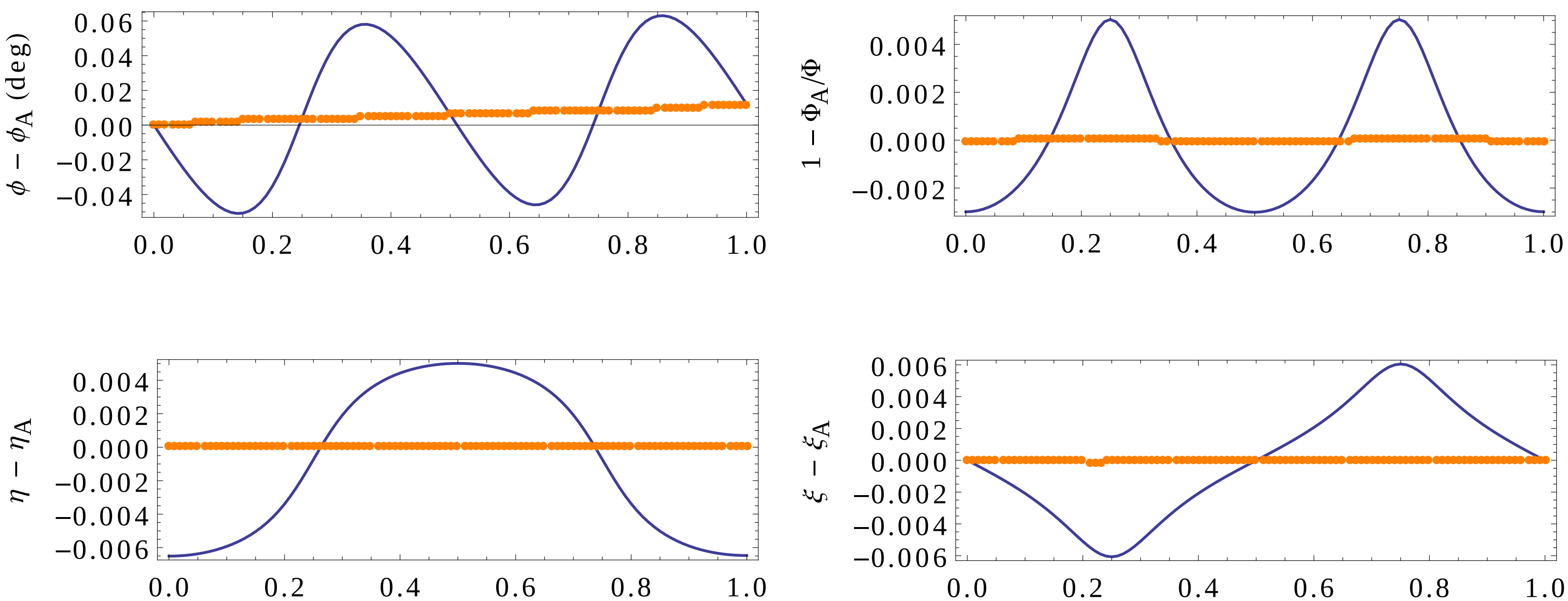


1:1 resonant DRO $a = d_y$

- Design parameters $a = 10 \Rightarrow \Phi' = 12.5$,

  $d_y = a \Rightarrow q'_0 = Q'_0 = 0, \qquad T_O = 6.24852$

  almost periodic: $|\boldsymbol{x}(T) - \boldsymbol{x}(0)| \sim |\boldsymbol{X}(T) - \boldsymbol{X}(0)| = 10^{-3}$



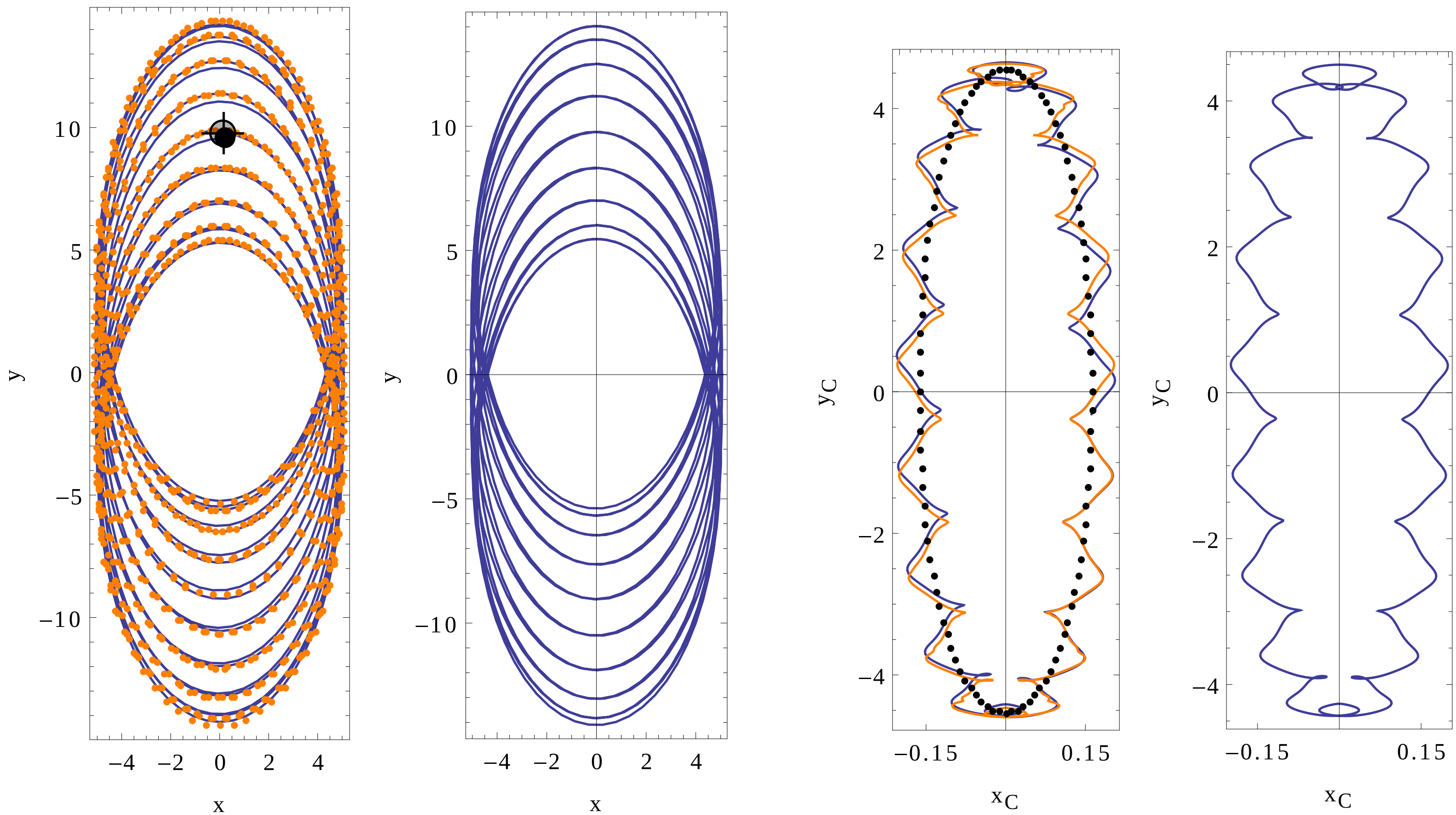
- Periodicity improved by differential corr. $\sim 10^{-13}$

Multiple resonant DRO: $a = 10$, $d_y = 5$

- Design parameters $a = 10 \Rightarrow \Phi' = 12.5$, $\Omega \approx \omega/20$

  $d_y = 5 \Rightarrow$, $q'_0 = 0$ $Q'_0 = 0.141645$,

  - but $\rho \equiv T_L/T_O = 18.3\ldots$ non-periodic
- Find Δa such that ρ integer/rational, by iterations
  - d_y, $q'_0 = 0$, fixed
  - $\rho = T_L(Q'_0(\Phi'), \Phi')/T_O(Q'_0(\Phi'), \Phi') = \rho(\Phi')$
- Secant method $a = 9.87661$, $T_L = 112.379$, $\rho = 18$
  - exactly periodic in mean elements
  - almost periodic: $|\boldsymbol{x}(T) - \boldsymbol{x}(0)| \sim |\boldsymbol{X}(T) - \boldsymbol{X}(0)| = 10^{-1}$
  - periodicity improved by differential corr. $\mathcal{O}(10^{-13})$


10
5
0
−5
−10
y
−4
−2
0
2
4
x
10
5
0
−5
−10
y
−4
−2
0
2
4
x
4
2
0
−2
−4
y_C
−0.15
0.15
x_C
4
2
0
−2
−4
y_C
−0.15
0.15
x_C

Conclusions

- Useful algorithm for computing DRO
  - low and higher order analytical solutions
  - particular case of co-orbital motion
- Low order solution discloses the nature of the DRO problem
  - provides orbit design parameters
    - dimension of the reference ellipse
    - minimum y distance to the primary
    - orbital and libration periods
- Higher order improves accuracy for large librating orbits
- Planar Hill problem chosen as a demonstration model
  - 3D case also feasible (in progress)
- Same techniques apply to the R3BP (in progress)