Efficient design of low lunar orbits based on Kaula recursions

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Background: Preliminary design

- Simplified models that comprise the bulk of the dynamics
- Gravitational potential: long-period oscillations $\gg$ short-p.
  - use just the few more relevant zonals in the initial steps
- Long-term evolution: remove the higher freq. of the motion to speed the process of mission design
- Moon: lumpy potential, design of low altitude lunar orbits needs to deal with tens of zonal harmonics
  - analytical approach: handle huge expressions formally
  - compact recursions in the literature (averaged flow)
- Correct computation of iicc: mean $\longleftrightarrow$ osculating needed
  - crucial for the design of unstable orbital configurations
  - case of science orbits under 3-body perturbations
- Need to improve existing formulas in the literature

Outline

- Recall Kaula's formulation of the zonal potential
- Orbit evolution: mean elements. Need mean $\overset{T}{\longleftrightarrow}$ osculating
  - T: periodic corr. derived from a generating function
- New T based on Kaula formulas
  - performance clearly surpass existing proposals
- Analytical design of low lunar orbits
  - compact and efficient with Kaula & Kaula-type formulas
  - global diagrams of frozen orbits
  - local eccentricity vector diagrams
- Conclusions

The Zonal potential

- Keplerian term $-\mu/r$ plus the disturbing potential $\mathcal{U} = -(\mu/r)\sum_{n\geq 2}(R_\oplus/r)^n C_{n,0} P_{n,0}(\sin\varphi)$
  - $\mu, R_\oplus, C_{n,0}$ model parameters, r, λ, ϕ spherical coordinates
  - axial symmetry: $\mathcal{U} = \mathcal{U}(r, -, \phi)$
- Kaula style: formulate $\mathcal{U}$ in orbital elements $(a, e, I, \Omega, \omega, M)$
  - true anomaly $f = f(M, e)$ instead of the mean one
  - $r = p/(1 + e\cos f)$, $p = a\eta^2$, $\eta = (1 - e^2)^{1/2}$
- Thus, $\mathcal{U} = -(\mu/a)(a/r)^2\eta\sum_{i\geq 2} V_i(a, e, I, -, \omega, M)$ with

$$V_i = \frac{R_\oplus^i}{a^i}\frac{C_{i,0}}{\eta^{2i-1}}\sum_{j=0}^{i}\mathcal{F}_{i,j}(I)\sum_{k=0}^{i-1}\binom{i-1}{k}e^k\cos^k f\cos[(i-2j)(f+\omega)-\pi_i]$$

  - $\pi_i = (i \bmod 2)\frac{\pi}{2}$ is the parity correction
  - $\mathcal{F}_{i,j}$ Kaula inclination functions (recursions in literature)

Long-term effects: mean elements

- Transformation $(a, e, I, \Omega, \omega, M) \xrightarrow{T} (a', e', I', \Omega', \omega', M'; \epsilon)$
  - primes: mean elements, $\epsilon \ll 1$ small parameter
- $\mathcal{U} \circ T = \sum_{i=1}^{m} (\epsilon^i / i!) U_i(a', e', I', -, \omega', -) + \mathcal{O}(\epsilon^{m+1})$
  - Taylor series in mean elements
  - short-period effects removed up to the truncation order
- T from a generating function $W = \sum_{i \geq 0} (\epsilon^i / i!) W_{i+1}$
  - finding T is the non-trivial subject of *perturbation theory*
- 1st order:
  - $U_1 = \langle \mathcal{U} \rangle_M = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{U} \, \mathrm{d}M$,
  - $W_1 = \frac{1}{n} \int (\mathcal{U} - U_1) \, \mathrm{d}M$, $\quad n$: mean motion
  - closed form integration $\mathrm{d}M = r^2 / (a^2 \eta) \mathrm{d}f$

- Low order corrections given by different authors

$$
\begin{aligned}
a - a' &= -\frac{2}{an}\frac{\partial W_1}{\partial M} \\
e - e' &= \frac{\eta}{ea^2 n}\left(\frac{\partial W_1}{\partial \omega} - \eta\frac{\partial W_1}{\partial M}\right) \\
I - I' &= -\frac{c}{a^2 n s \eta}\frac{\partial W_1}{\partial \omega} \\
\Omega - \Omega' &= -\frac{1}{a^2 n s \eta}\frac{\partial W_1}{\partial I} \\
\omega - \omega' &= \frac{1}{a^2 n \eta}\left(\frac{c}{s}\frac{\partial W_1}{\partial I} - \frac{\eta^2}{e}\frac{\partial W_1}{\partial e}\right) \\
M - M' &= \frac{1}{a^2 n}\left(2a\frac{\partial W_1}{\partial a} + \frac{\eta^2}{e}\frac{\partial W_1}{\partial e}\right)
\end{aligned}
$$

- Corrections in non-singular elements are also available

The long-term (averaged) potential

- $U_1 = -\frac{\mu}{a}\sum_{i\geq 2}\langle V_i\rangle_f$, average terms $\cos^k f \cos[(i-2j)(f+\omega)-\pi_i]$
  - expand $\cos[(i-2j)(f+\omega)-\pi_i]$, then

$$\cos^k f \left\{ \begin{array}{l} \cos(i-2j)f \\ \sin(i-2j)f \end{array} \right\} = \frac{1}{2^k}\sum_{l=0}^{k}\binom{k}{k-l}\left\{ \begin{array}{l} \cos(i-2j-k+2l)f \\ \sin(i-2j-k+2l)f \end{array} \right\}$$

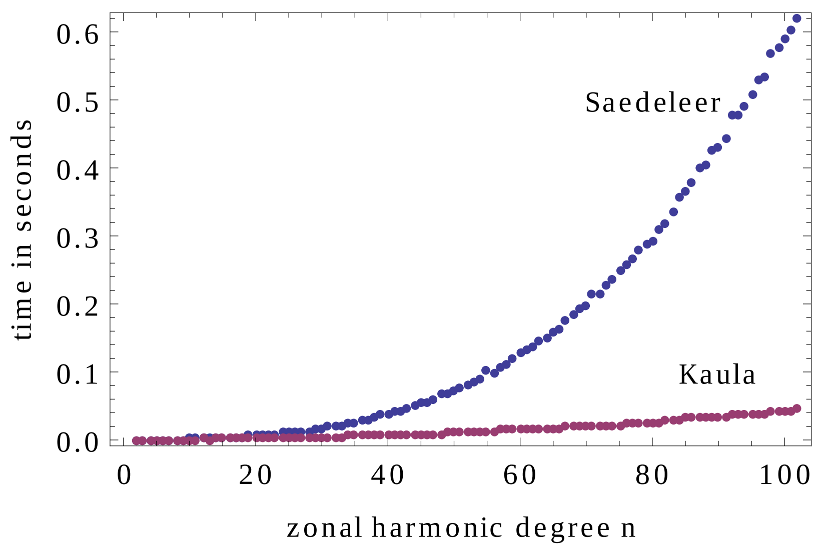
  - free from f if $(i-2j-k+2l)=0$

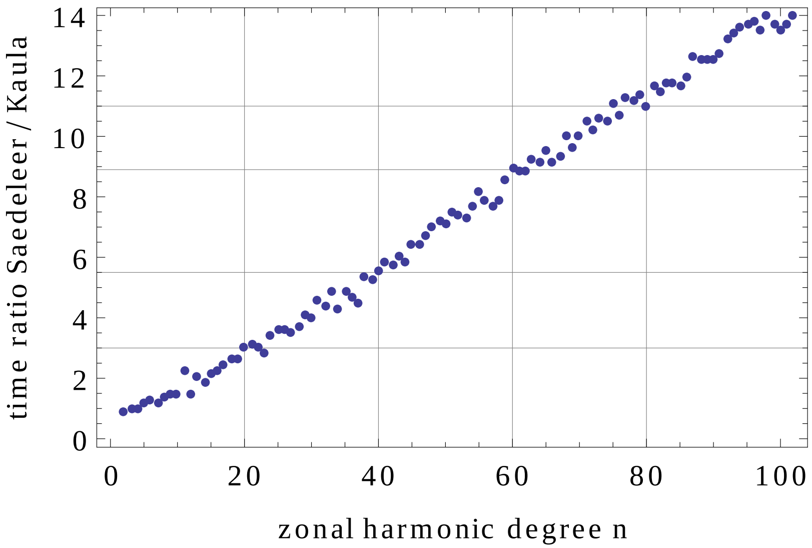
- $\langle V_i\rangle_f = \eta\frac{R_\oplus^i}{a^i}C_{i,0}\sum_{j=0}^{i}\mathcal{F}_{i,j}(\sin I)\mathcal{G}_{i,j}(e)\cos[(i-2j)\omega-\pi_i]$
  - Kaula eccentricity functions

$$\mathcal{G}_{i,j} = \frac{1}{(1-e^2)^i}\sum_{l=0}^{\tilde{\jmath}-1}\binom{i-1}{q}\binom{q}{l}\frac{e^q}{2^q}, \qquad q = 2l+i-2\tilde{\jmath}$$

  - either $\tilde{\jmath}=j$ when $i\geq 2j$, or $\tilde{\jmath}=i-j$ when $i<2j$

- Performance comparison with alternative formulas
  - Kaula, Theory of satellite geodesy, 1966
    - $\sim$ quadratic grow with the number of harmonics
  - De Saedeleer, Celest. Mech. Dyn. Astron. 91, 2005
    - $\sim$ cubic grow with the number of harmonics





- Ratio grows **linearly** with the number of harmonics

Frozen orbits

- Long-term potential free from M and Ω
  - a, $N = \sqrt{\mu a}\eta \cos I$ constant $\Rightarrow I = I(e; a, N)$
- Evolution eqs. for e and ω, or $k = e\cos\omega$ and $q = e\sin\omega$

$$\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{1}{na^2}\left[\eta\left(\frac{1}{e}\frac{\partial U_1}{\partial\omega}\right)\cos\omega - \left(e\frac{\partial U_1}{\partial\eta} - \eta\frac{\partial U_1}{\partial e} + \frac{e}{\eta}\frac{c^2}{s}\frac{\partial U_1}{\partial s}\right)\sin\omega\right]$$

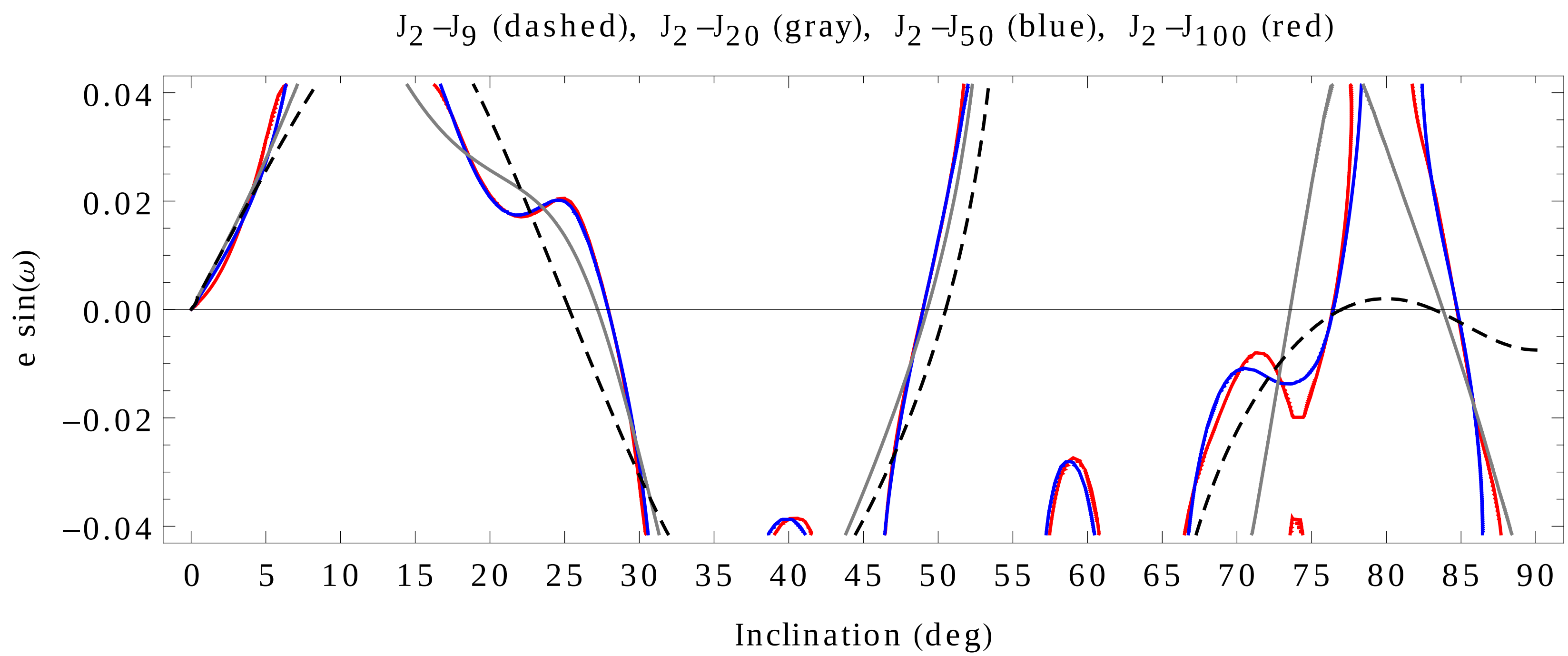
$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{1}{na^2}\left[\eta\left(\frac{1}{e}\frac{\partial U_1}{\partial\omega}\right)\sin\omega + \left(e\frac{\partial U_1}{\partial\eta} - \eta\frac{\partial U_1}{\partial e} + \frac{e}{\eta}\frac{c^2}{s}\frac{\partial U_1}{\partial s}\right)\cos\omega\right]$$

- $\omega = \pm\pi/2 \quad\Rightarrow\quad \frac{\mathrm{d}q}{\mathrm{d}t} = \mp\frac{\eta}{na^2}\left(\frac{1}{e}\frac{\partial U_1}{\partial\omega}\right)\Big|_{\omega=\pm\frac{\pi}{2}} = 0$
- Constraint eq. for the frozen orbit: $\frac{\mathrm{d}k}{\mathrm{d}t} = \mp\left(e\frac{\mathrm{d}\omega}{\mathrm{d}t}\right)\Big|_{\omega=\pm\frac{\pi}{2}} = 0$
  - $\dot{\omega}(a_0, e, I, -, \omega = \pm\frac{\pi}{2}, -) = 0$
  - formulas in the literature for any number of zonals
  - global diagrams (I, e) of frozen orbits (for a given $a = a_0$)

- Energy integral $E = -\mu/(2a) + U_1(e, \omega; a, N)$
  - eccentricity vector diagrams ($k = e \cos \omega, q = e \sin \omega$)
  - simple contour plots of E in the parameters plane (a, N)
    - easily implemented using
      $e^m \cos m\omega = \frac{1}{2}[(k - \mathbb{i}\, q)^m + (k + \mathbb{i}\, q)^m]$
      $e^m \sin m\omega = \frac{\mathbb{i}}{2}[(k - \mathbb{i}\, q)^m - (k + \mathbb{i}\, q)^m]$, with m integer

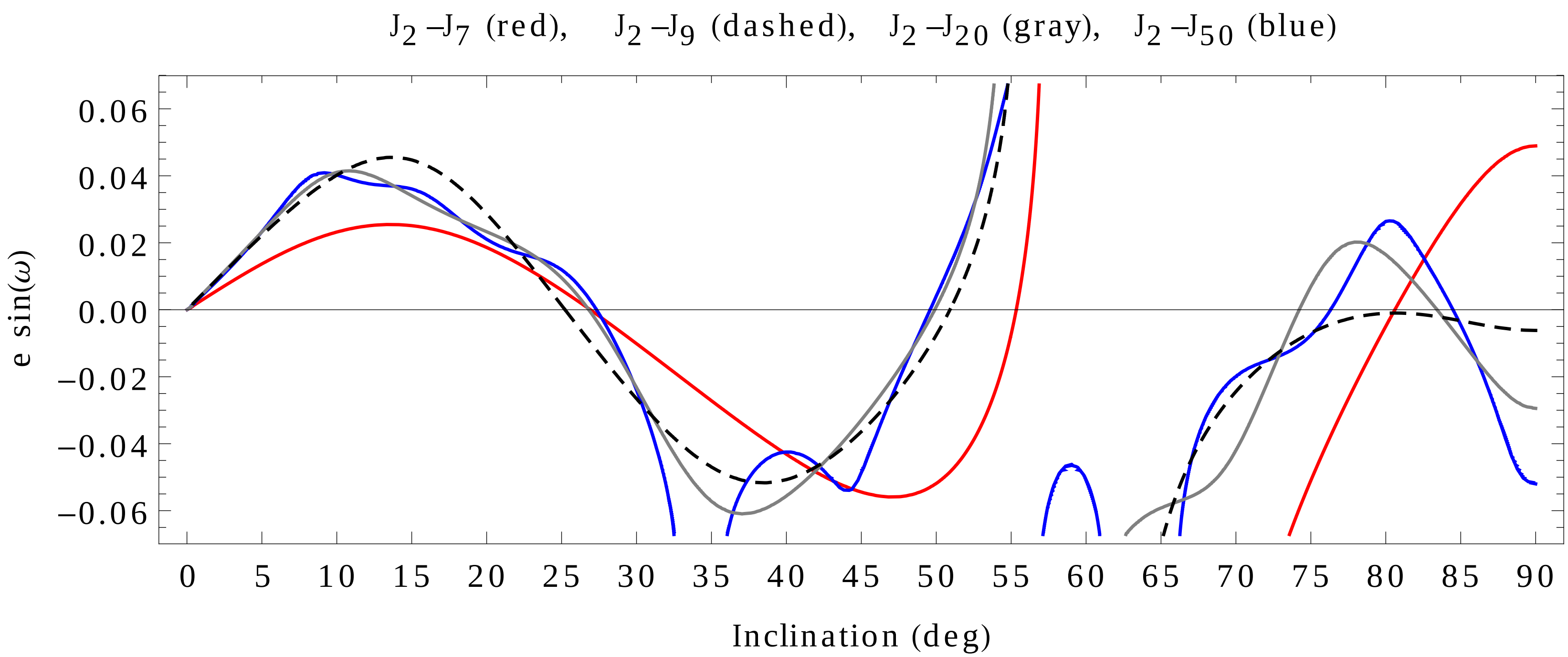
Design of low-lunar orbits

- I-e diagram of frozen orbits for different zonal truncations



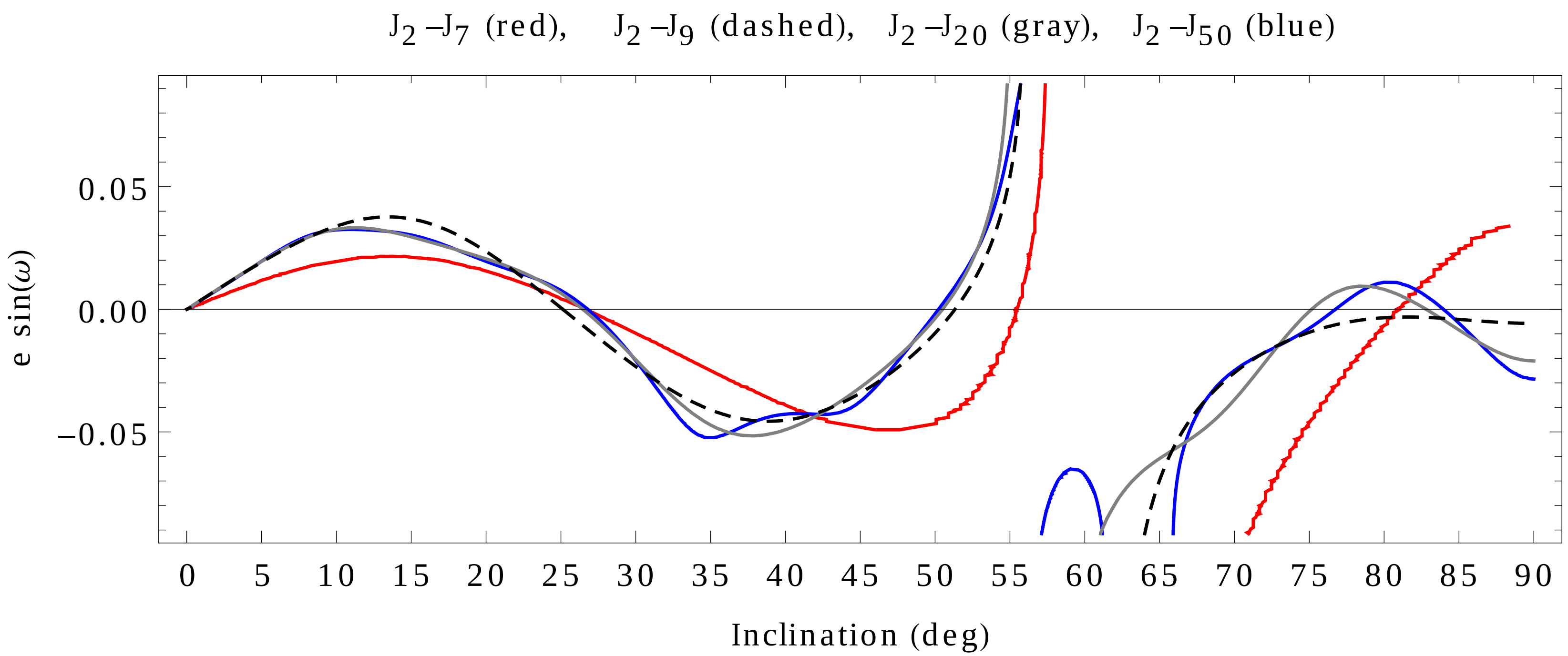
- $a = 1813$ km $\sim$ 75 km over the surface of the moon

- I-e diagram of frozen orbits for different zonal truncations



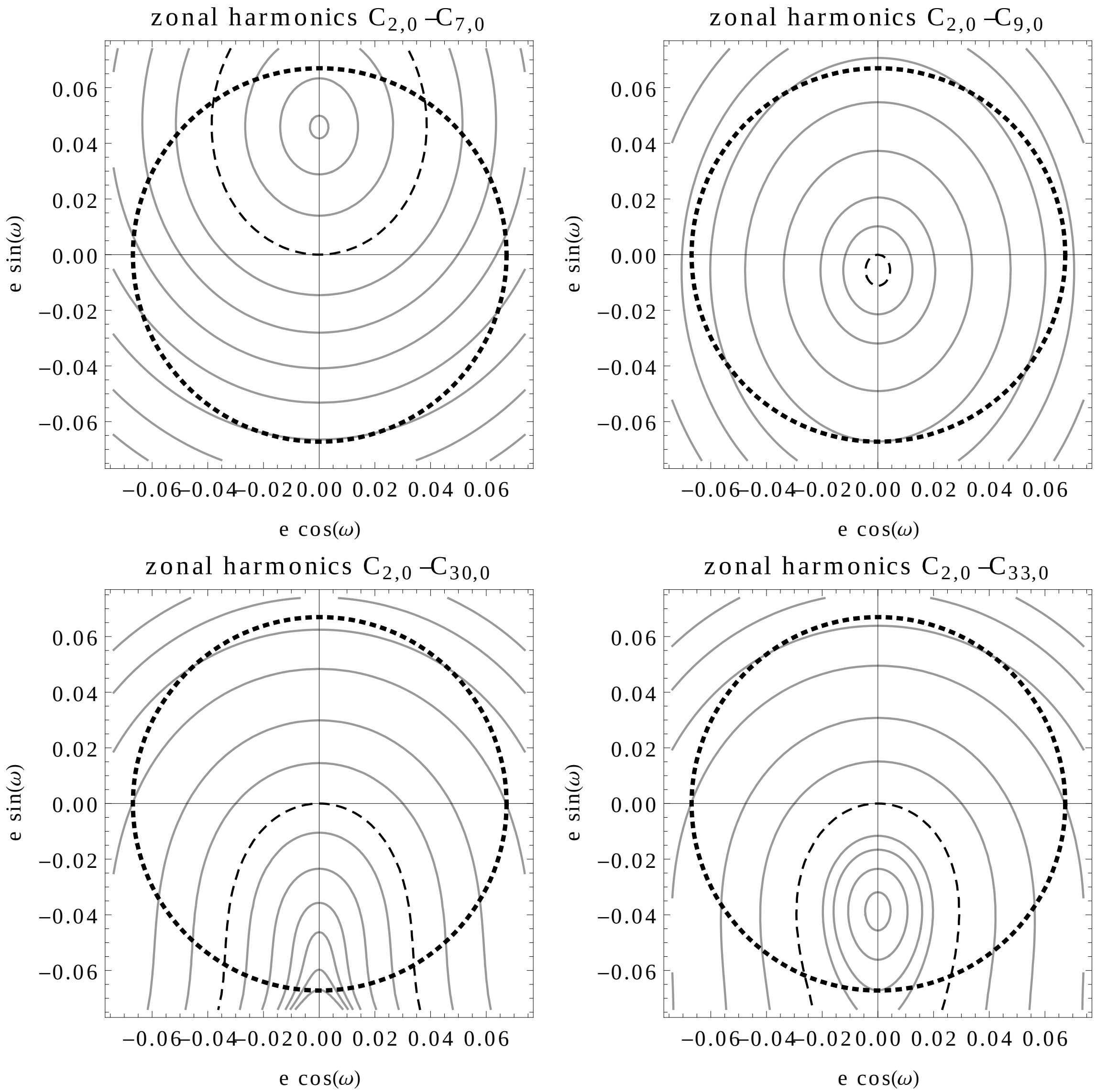
- $a = 1863$ km $\sim$ 125 km over the surface of the moon

- I-e diagram of frozen orbits for different zonal truncations



- $a = 1913$ km $\sim$ 175 km over the surface of the moon

Eccentricity vector diagrams: ~ 125 km altitude



Conclusions

- Mission design of low lunar orbits need full potential models
  - 50 truncation, at least, recommended for science orbits
- Full models are efficiently handled analytically
  - Kaula 1960's expressions much more efficient that new proposals in the literature
- Long-term dynamics without need of numerical integration
  - e vector diag., contour plots of the averaged potential
  - (I, e) diag., from the frozen orbit constraint equation