EFFICIENT DESIGN OF LOW LUNAR ORBITS BASED ON KAULA RECURSIONS

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ABSTRACT

It is shown that the potential of the zonal problem of artificial satellite theory, up to an arbitrary degree, is more efficiently formulated using Kaula linear perturbation theory than other compact alternatives in the literature. Construction of the evolution equations based on this formulation is clearly more advantageous than the direct, brut force approach using symbolic algebra systems, and is applied to the design of low lunar orbits.

***Index Terms*—** Gravitational potential, mean elements equations, averaging, Kaula recursions, perturbation theory

1. INTRODUCTION

Preliminary design of artificial satellite missions commonly relays on the use of simplified models that comprise the bulk of the dynamics. For orbits about massive bodies, the effects of the non-centralities of the gravitational potential in the orbital parameters are traditionally decomposed into secular and periodic terms, the later comprising both short- and long-period oscillations. Because the amplitude of the long-period oscillations is roughly one order of magnitude larger than the short-period effects, dealing with just the few more relevant zonal harmonics of the potential is generally suitable for the initial steps of mission designing of artificial satellites. In addition, the long-term evolution of the orbital parameters is customarily investigated through averaging procedures that remove the higher frequencies of the motion, in this way notably speeding the computations.

Explicit expressions are customarily used when the dynamics can be represented by lower degree truncations of the gravitational potential [1, 2, 3, 4, 5]. However, there are cases in which the use of simplified models is not an option and full zonal potential models must be used instead. The paradigm is provided by the moon, where, due to the irregular character of the moon gravity field, mission designing of low altitude lunar orbits needs to deal with tens of, contrary to just a few, zonal harmonics [6]. The analytical approach is still possible [7, 8], but the requirement of handling formally huge expressions usually discourages mission planners, who rather resort to numerical procedures. Still, useful compact formulas for dealing analytically with this problem exist in the literature since many years ago [9], yet for practical application they are commonly limited to the equations of the averaged flow, which, besides, are particularized for the case of low eccentricity orbits [10, 11, 12, 13].

Based on Kaula's popular work [14], we re-derive the long-term potential of the zonal problem in closed form and show that Kaula's approach in orbital elements provides much more efficient formulas for the construction of the mean elements potential that recent alternative proposals in the literature [15, 16]. The necessity of having available efficient expressions for the long-term zonal potential, from which the evolution equations of the orbit are directly derived, is illustrated with application to the design of low lunar orbits.

2. THE ZONAL POTENTIAL

The gravity potential is customarily given as a sum of trigonometric functions of the latitude and longitude whose coefficients involve zonal, sectorial and tesseral harmonics, as well as inverse powers of the radius (see [17], for instance). However, except for particular resonances of the satellite's mean motion with the rotation rate of the attracting body, the effects of tesseral perturbations average out. In that case, the zonal potential comprises the Keplerian term $-\mu/r$ and the disturbing function

$$\mathcal{U} = -\frac{\mu}{r}\sum_{n\geq 2}\frac{R_\oplus^n}{r^n}C_{n,0}P_{n,0}(\sin\varphi). \tag{1}$$

where r, and φ stand for radius and geocentric latitude, respectively; $P_{n,0}$ are Legendre polynomials, and $C_{n,0} = -J_n$ are zonal harmonic coefficients. In addition to the harmonic coefficients, the values of the gravitational parameter μ and the equatorial radius of the central body $R_\oplus$ are what define a gravitational model.

For orbital mechanics problems it is useful to write Eq. (1) in orbital elements, viz. $(a, e, I, \Omega, \omega, M)$ for semi-major axis, eccentricity, inclination, right ascension of the ascending node, argument of the periapsis, and mean anomaly,

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respectively. We recall that $\sin\varphi = \sin I \sin\theta$, where the argument of latitude is $\theta = f + \omega$, and the true anomaly f is an implicit function of the mean anomaly M.

Following Kaula's approach [14], we replace the radius by

$$r = p/(1 + e\cos f), \tag{2}$$

in which p is the conic parameter, given by

$$p = a\eta^2, \tag{3}$$

and

$$\eta = \sqrt{1 - e^2}, \tag{4}$$

is customarily known as the eccentricity function. Then, Eq. (1) is written in the form[1]

$$\mathcal{U} = -\frac{\mu}{a}\left(\frac{a}{r}\right)^2 \eta \sum_{i\geq 2} V_i(a, e, I, -, \omega, M), \tag{5}$$

in which

$$\begin{aligned} V_i &= \frac{R_\oplus^i}{a^i}\frac{C_{i,0}}{\eta^{2i-1}}\sum_{j=0}^{i}\mathcal{F}_{i,j}(I)\sum_{k=0}^{i-1}\binom{i-1}{k} \\ &\quad \times e^k \cos^k f \cos[(i-2j)(f+\omega) - \pi_i], \end{aligned} \tag{6}$$

where

$$\pi_i = (i \bmod 2)\frac{\pi}{2} \tag{7}$$

is the parity correction, and $\mathcal{F}_{i,j}$ notes Kaula's inclination functions particularized for the case of the zonal problem. That is,

$$\begin{aligned} \mathcal{F}_{i,j} &= \sum_{l=0}^{\min(j,\lfloor i/2\rfloor)} \frac{(-1)^{j-l-i_0}}{2^{2i-2l}}\frac{(2i-2l)!}{l!(i-l)!(i-2l)!} \\ &\quad \times \binom{i-2l}{j-l}\sin^{i-2l} I, \quad i \geq 2l. \end{aligned} \tag{8}$$

where $\lfloor i/2 \rfloor$ denotes an integer division. Remarkably, the inclination functions can be evaluated by means of recursion formulas [18, 19].

To abbreviate notation in what follows we replace $s \equiv \sin I$, $c \equiv \cos I$, and adhere to the index notation in [20], namely

$$i^\star = i \bmod 2, \quad i_j = \lfloor (i-j)/2 \rfloor, \quad i_j^\star = i_j + i^\star. \tag{9}$$

3. LONG-TERM EFFECTS IN MEAN ELEMENTS

Contrary to ephemeris computation, which must be supplied in osculating elements, the orbit evolution is customarily studied in mean elements. That is, a transformation of variables $(a, e, I, \Omega, \omega, M) \xrightarrow{T} (a', e', I', \Omega', \omega', M'; \epsilon)$, in which $\epsilon \ll 1$ notes the small parameter of the transformation, and the prime variables denote mean elements, is carried out such that the short-period variations, in the new variables, are removed from Eq. (5) up to some truncation order m of the Taylor series expansion. Thus, neglecting terms of $\mathcal{O}(\epsilon^{m+1})$ and higher, it is obtained

$$\mathcal{U} \circ T = \sum_{i=1}^{m} \frac{\epsilon^i}{i!} U_i(a', e', I', -, \omega' -). \tag{10}$$

For conservative problems, like the current case, the transformation T is derived from a generating function $W = \sum_{i\geq 0}(\epsilon^i/i!)W_{i+1}$, whose computation is the nontrivial subject of perturbation theory (see [21], for instance). However, up to the first order of ϵ, the mean/osculating transformation is easily computed as follows.

First, U_1 is chosen as the average of the disturbing function $\mathcal{U}$ over the mean anomaly,

$$U_1 = \langle \mathcal{U} \rangle_M = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{U}\,\mathrm{d}M, \tag{11}$$

Next, W_1 is computed as

$$W_1 = \frac{1}{n}\int (\mathcal{U} - U_1)\,\mathrm{d}M, \tag{12}$$

where $n = \sqrt{\mu/a^3}$ is the mean motion. Then, up to the first order of ϵ, the transformation is given by the corrections [2]

$$\begin{aligned} a - a' &= -\frac{2}{an}\frac{\partial W_1}{\partial M} \\ e - e' &= \frac{\eta}{ea^2 n}\left(\frac{\partial W_1}{\partial \omega} - \eta\frac{\partial W_1}{\partial M}\right) \\ I - I' &= -\frac{c}{a^2 ns\eta}\frac{\partial W_1}{\partial \omega} \\ \Omega - \Omega' &= -\frac{1}{a^2 ns\eta}\frac{\partial W_1}{\partial I} \\ \omega - \omega' &= \frac{1}{a^2 n\eta}\left(\frac{c}{s}\frac{\partial W_1}{\partial I} - \frac{\eta^2}{e}\frac{\partial W_1}{\partial e}\right) \\ M - M' &= \frac{1}{a^2 n}\left(2a\frac{\partial W_1}{\partial a} + \frac{\eta^2}{e}\frac{\partial W_1}{\partial e}\right) \end{aligned}$$

where $\partial W_1/\partial M = (a/r)^2\eta\,\partial W_1/\partial f$ and, after evaluation, the right members are written in prime variables when moving from mean to osculating elements (direct transformation), and in osculating (non primed) variables when moving from osculating to mean elements (inverse transformation). This transformation is affected of singularities due to the set of elements chosen, but it can be easily reformulated in non-singular elements when required [22, 23].

To avoid expansions of the elliptic motion when solving the quadratures in Eqs. (11) and (12), the differential relation

[1]The reasons for keeping the factor $(a/r)^2\eta$ out of the summation will be apparent later.

$\mathrm{d}M = r^2/(a^2\eta)\,\mathrm{d}f$ is customarily used.[2] Then, replacing Eq. (5) into Eq. (11),

$$U_1 = -\frac{\mu}{a}\sum_{i\geq 2}\langle V_i\rangle_f. \tag{13}$$

Terms $\langle V_i\rangle_f$ are more easily computed by expanding Eq. (6) as a Fourier series in f. This is done by rearranging terms $\cos^k f\cos[m(f+\omega)-\pi_i]$, with $m=(i-2j)$, in Eq. (6). First, $\cos[m(f+\omega)-\pi_i]$ is written

$$\begin{aligned}\cos[m(f+\omega)-\pi_i] &= \cos mf\cos(m\omega-\pi_i)\\ &\quad -\sin mf\sin(m\omega-\pi_i).\end{aligned} \tag{14}$$

Then, using the standard trigonometric reductions

$$\begin{aligned}\cos\beta\cos^k\alpha &= \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}\cos^{k-1}\alpha,\\ \sin\beta\cos^k\alpha &= \frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2}\cos^{k-1}\alpha,\end{aligned} \tag{15}$$

one easily arrives to

$$\begin{aligned}\cos^k f\left\{\begin{array}{c}\cos(i-2j)f\\ \sin(i-2j)f\end{array}\right\} &= \frac{1}{2^k}\sum_{l=0}^{k}\binom{k}{k-l}\\ &\quad\times\left\{\begin{array}{c}\cos(i-2j-k+2l)f\\ \sin(i-2j-k+2l)f\end{array}\right\},\end{aligned} \tag{16}$$

which shows that the only terms of Eq. (6) that are free from f come from those terms of Eq. (16) such that $l=\frac{1}{2}[k-(i-2j)]$ or, equivalently, $k-l=\frac{1}{2}(k+i)-j$. Hence,

$$\langle V_i\rangle_f = \eta\frac{R_\oplus^i}{a^i}C_{i,0}\sum_{j=0}^{i}\mathcal{F}_{i,j}(s)\mathcal{G}_{i,j}(e)\cos[(i-2j)\omega-\pi_i], \tag{17}$$

where

$$\mathcal{G}_{i,j} = \frac{1}{(1-e^2)^i}\sum_{k=0}^{i-1}\binom{i-1}{k}e^k\frac{1}{2^k}\binom{k}{\frac{k+i}{2}-j}. \tag{18}$$

As expected, except for the factor η that we left out of the summation in Eq. (17), the functions $\mathcal{G}_{i,j}$ in Eq. (18) are no more than Kaula eccentricity functions for the particular case of the zonal problem. Indeed, because the summation index l in Eq. (16) is integer, $k+i$ must be even in Eq. (18), which, therefore, can be rearranged in the efficient form proposed by Kaula

$$\mathcal{G}_{i,j} = \frac{1}{(1-e^2)^i}\sum_{l=0}^{\tilde{j}-1}\binom{i-1}{q}\binom{q}{l}\frac{e^q}{2^q}, \tag{19}$$

where $q=2l+i-2\tilde{j}$, and either $\tilde{j}=j$ when $i\geq 2j$, or $\tilde{j}=i-j$ when $i<2j$, cf. Eq. 3.66 of [14].

[2] This fact gives sense to the factor out of the summation in Eq. (5).

4. PERFORMANCE COMPARISONS

More than 40 years later than the seminal work of Kaula, alternative expressions have being provided from the point of view of Hamiltonian perturbation theory [15]. However, while both sets of formulas must be obviously equivalent when expanded, we show here that Kaula's arrangement of the summations notably eases the construction of the averaged potential when the number of zonal harmonics considered goes beyond the first few terms.

Indeed, as shown in Fig. 1, the time spent in the explicit construction of the averaged potential up to a given degree n is roughly the same either with Kaula's or Saedeleer's expressions for the lower degrees of the zonal potential. However, when the degree starts to grow, we find that the time spent when using Saedeleer's expressions clearly increases over the time needed when using Kaula's expressions, at an almost constant rate that is approximately proportional to 1 tenth of the zonal harmonic degree n. Thus, in Saedeleer's approach, the time spent in constructing the zonal term of 20th degree is ~ 3 times longer than the time needed when using Kaula's formulas, ~ 5.5 times longer for the zonal term of 40th degree, ~ 9 times longer for the zonal term of 60th degree, or ~ 11 times longer for the zonal term of 80th degree.

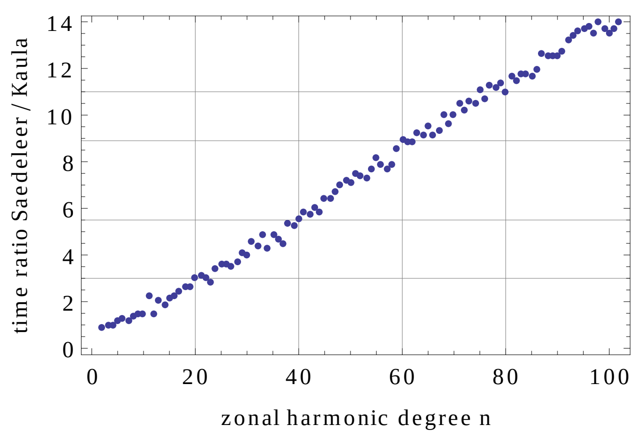

Fig. 1. Performance of Kaula expressions in the construction of the mean elements potential relative to analogous expressions in [15].

The computations have been carried out with Wolfram *Mathematica* 9 running under macOS High Sierra, version 10.13.6, with a 2.8 GHz Intel Core i7 processor and 16 GB of RAM. Absolute times using each approach are depicted in Fig. 2. We checked that while the computing time grows with the degree in a cubic rate when using Saedeleer's approach, it just grows only slightly higher than quadratic when using Kaula's recursions.

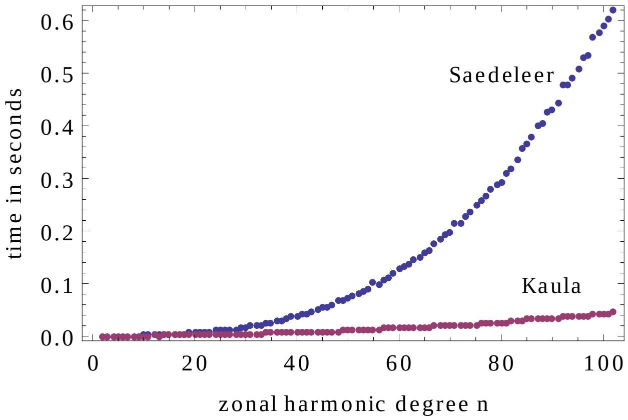

Fig. 2. Time spent in the computation of each term of the mean elements zonal potential with the different algorithms.

5. DESIGN OF LOW LUNAR ORBITS

Because of the lumpy character of the moon's gravity field, the use of full zonal potential models is mandatory even in the preliminary steps of mission designing of low lunar orbits. In particular, the choice of lower degree zonal potentials not only provides unacceptable results from a quantitative point of view, but it my provide also wrong qualitative results, as it has been pointed out in [24]. Exploring the sensitivity of the orbit design problem with respect to the truncation degree of the zonal potential is definitely facilitated using general expressions as those of Kaula in Eq. (13), cf. [20].

In this section we illustrate the utility of having analytical expressions of the form of Eq. (13), which is valid for an arbitrary degree, to explore the dynamics of low lunar orbits.

First of all, we note that the satellite motion under the only action of the zonal potential is a conservative problem. Therefore, it accepts the energy integral $E = T + V$, where T is the kinetic energy and $V = -\mu/r + \mathcal{U}$ is the potential energy. Hence,

$$E = -\frac{\mu}{2a} - \frac{\mu}{a}\left(\frac{a}{r}\right)^2 \eta \sum_{i\geq 2} V_i(a, e, I, -, \omega, M) \qquad (20)$$

as follows from Eq. (5). Besides, the problem enjoys axial symmetry, which is the reason why the right ascension of the ascending node Ω is missing in the potential. In consequence, in addition to the energy, the zonal problem admits the polar component of the angular momentum

$$H = \sqrt{\mu a(1-e^2)}\cos I, \qquad (21)$$

as integral. The effect of this integral is to decouple the variation of the node from the other variation equations, in which, besides, the variation of $I = I(e, a; H)$ does not need to be integrated, thus making the zonal problem of just 2 degrees of freedom.

On the other hand, the zonal problem is also conservative in mean elements, so it also admits the energy integral

$$E' = -\frac{\mu}{2a'} - \frac{\mu}{a'} \sum_{i\geq 2} \langle V_i \rangle_f (a', e', I', -, \omega', -), \qquad (22)$$

as well as

$$H' = \sqrt{\mu a'(1-e'^2)}\cos I'. \qquad (23)$$

But now, because of the averaging process, the *mean* mean anomaly M' is absent in the right member of the equations of motion. This fact converts the mean semi-major axis a' into an integral of the mean elements motion. The effect of this new integral is to decouple the mean anomaly evolution form the variation of the other elements. In consequence, solving the zonal problem in mean elements is reduced to the integration of the variation equations of e' and ω'. That is, a conservative one degree of freedom problem, which, therefore makes the zonal problem in mean elements integrable.

Instead of trying to compute the analytical solution, the integral H' can be used to rearrange the terms given by Eq. (17) of the mean elements potential in Eq. (13), in the form $\langle V_i \rangle_f \equiv \langle V_i \rangle_f (a', e', I'(e'; a', H'), \omega')$. Hence, Eq. (13) is written as

$$U_1 = -\frac{\mu}{a'} \sum_{i\geq 2} \langle V_i \rangle_f (e', \omega'; a', H') = E' + \frac{\mu}{a'} = \text{constant}.$$

Therefore, the flow in mean elements can be represented by contour plots of the constant, mean elements potential, in the parameters plane (a', H').

Alternatively, instead of using H' it is customary to use the inclination of the circular orbits $\sigma = \sigma(a', H')$ as one of the parameters because it provides a higher insight into the mean elements problem. Indeed, $e' = 0$ for circular orbits, and hence

$$\sigma = \cos I'_{\text{circular}} = H'/\sqrt{\mu a'} = \text{constant}.$$

Therefore, we explore the dynamics of the mean elements zonal problem by fixing a' and $\cos I'_{\text{circular}}$, and plotting different contours $U_1 = \text{constant}$ for different pairs of initial conditions $e' = e'_0$, $\omega' = \omega'_0$. The representation of the mean elements flow in this way for different truncations of the zonal potential will reveal the number of zonal harmonics required in the preliminary design of low lunar orbits [20].

A sample illustration of this procedure is presented in Fig. 3, where, in order to focus on non-impact, low altitude orbits that are typical for science missions, the different contour plots are depicted in the $(e'\cos\omega', e'\sin\omega')$ representation. The parameters of a low altitude high inclination lunar orbit that remains, on average, 125 km over the surface of the moon, and with an inclination of 88 deg. for a circular orbit have been selected. The dotted circle in the plots of Fig. 3 marks the eccentricity limit for impact orbits, whereas the dashed contour corresponds to the energy manifold of circular orbits.

We see that predictions for a $C_{2,0}$–$C_{7,0}$ truncation of the lunar potential (top plot of Fig. 3), show that the manifold of circular orbits soon or later lead the orbiter to impact the surface of the moon. Still, surrounded by this manifold, there exists an orbit with constant eccentricity, on average, that remains with *frozen* argument of the periapsis, on average, $\omega' = \pi/2$. However, when the truncation of the mean zonal potential is extended to include up to $C_{9,0}$ (second plot of Fig. 3), the eccentricity of the circular orbit no longer grows enough to yield impact. Quite on the contrary, the energy manifold of the circular orbit surrounds quite closely a frozen orbit with very low eccentricity that, in this case, has the argument of the periapsis in the opposite direction of the previous case: $\omega' = -\pi/2$.

The evolution of these types of orbits keeps changing for truncations of the potential to increasing degrees, and when $C_{2,0}$–$C_{30,0}$ are taken into account (third plot of Fig. 3), the eccentricity of all the orbits of interest will end increasing to the limit of impact. Subsequent computations for increasing number of zonal harmonics show that the dynamics seems to stabilize for the $C_{2,0}$–$C_{33,0}$ truncation (bottom plot of Fig. 3), and it has been checked that including higher order harmonics in the mean elements equations only contribute small quantitative variations for an orbit of these characteristics.

6. CONCLUSIONS

The effect of long-term disturbances produced by the noncentralities of the gravitational potential on an artificial satellite of the moon can be efficiently scrutinized up to an arbitrary degree using Kaula equations particularized for the zonal problem in mean elemnts. On the other hand, the common case of orbits about earth-like bodies is sensitive to second order effects of the zonal harmonic of the second degree. In that case, Kaula's compact summations must be complemented with the explicit expressions of these second order effects. This work is in progress and results will be reported elsewhere.

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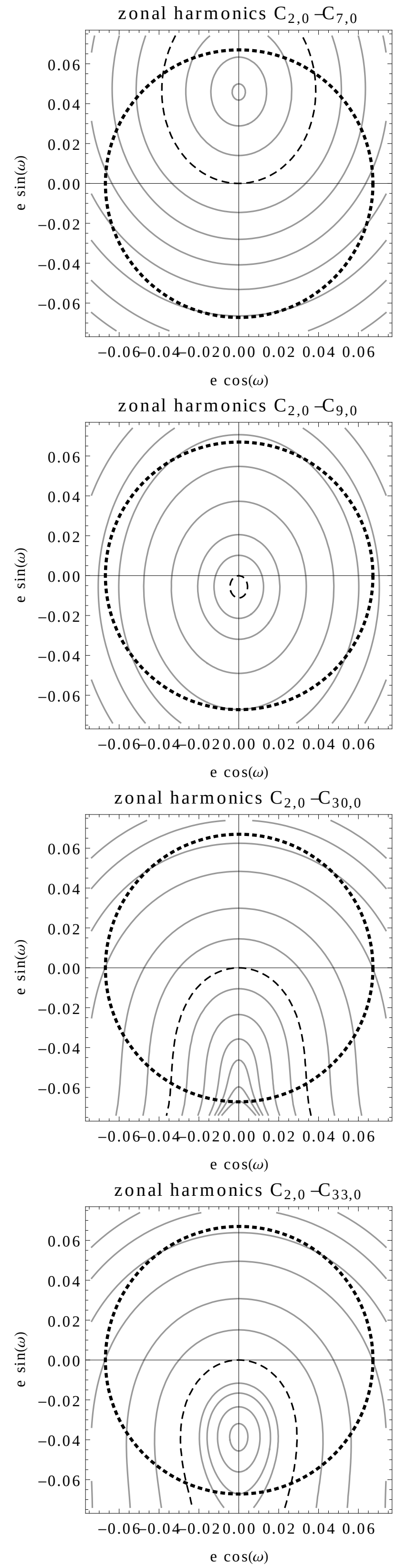

Fig. 3. Long-term dynamics of a lunar orbit with $a = R_\oplus +$ 125 km, $I_{\mathrm{circular}} = 88\,\mathrm{deg}$.

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