A FAST AN EFFICIENT ALGORITHM FOR THE COMPUTATION OF DISTANT RETROGRADE ORBITS

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ABSTRACT

The construction of an algorithm for the computation of distant retrograde orbits based on the use of Lindstedt series is discussed. The solution is split into the time history of the mean elements, which is useful in the description of the long-term dynamics, and a set of short-period corrections that provide reasonably accurate ephemeris. It is shown that the analytical solution reproduces quite well the behavior of typical distant retrograde orbits with small librations about the co-orbital primary, but it also captures the main features of the dynamics of orbits with large amplitude librations.

***Index Terms*—** Distant retrograde orbits, Hill's problem, perturbation theory, Lindstedt series, periodic orbits

1. INTRODUCTION

Details on the construction of an algorithm for the computation of distant retrograde orbits are presented. The algorithm is based on the computation of an approximate analytical solution of the restricted three body problem in the Hill problem approximation that provides accurate estimation of two basic design parameters. Notably, these parameters can be used for the computation of initial conditions of orbits that are periodic on average, and almost periodic in the original Hill problem. Following application of iterative differential corrections to the initial conditions predicted by the analytical solution makes them to easily converge to the initial conditions and period of a true periodic orbit with the characteristics fixed by the design parameters, which is a standard technique in preliminary mission design [1, 2, 3, 4].

The analytical solution consist roughly of a drifting ellipse, whose guiding center moves around the primary with long-period oscillations, and in which the linear growing of the phase of the satellite is modulated with long-period variations [5]. The analytical solution that feeds the algorithm splits into two parts of different nature. The first one provides the periodic corrections needed for converting osculating elements into the mean ones that describe the long- term evolution of the dynamics. The evolution equations of the mean elements as well as the short-period corrections are computed by usual Hamiltonian perturbation theory based on Lie transforms [6]. The second part of the analytical solution gives the time history of the mean elements in the form of five Lindstedt series that are needed for describing: the time scale in which the Lindstedt series evolve (1 series); the time evolution of the guiding center of the reference ellipse (2 series); the linear frequency with which the satellite evolves, on average (1 series); the long-period modulation of the phase of the satellite (1 series), cf. [7].

The use of the algorithm is illustrated with different examples, ranging from the typical case of 1:1-resonant distant retrograde orbits, in which the satellite remains always far away enough from the primary, to the challenging case of those higher order resonances in which the amplitude of the libration of the guiding center of the orbit can take the satellite much closer to the primary.

2. PERTURBATION MODEL AND METHOD

In the restricted three-body problem approximation, distant retrograde orbits (DRO) evolve about the smaller primary but out of its sphere of influence (see, [8, 9, 10, 11, 12], for instance). Hence, the relative motion of the orbiter about the smaller primary cannot be approached as the usual case perturbed Keplerian motion, typical of orbits about planetary satellites, in which the gravitation of the smaller primary dominates, and the gravitational pull of the massive primary is taken as a perturbation [13]. In fact, both the massless body and the smaller primary evolve about the massive primary with co-orbital motion. Because of that, DROs are commonly called quasi-satellite orbits.

We constrain ourselves to the planar case of the Hill problem model, which is formulated in epicyclic canonical variables (ϕ, q, Φ, Q), viz.

$$\Phi = \frac{1}{2\omega}\left[(X + y\omega)^2 + (2Y + x\omega)^2\right], \qquad (1)$$

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$$\sin\phi = -(2Y + x\omega)/\sqrt{2\Phi}, \quad (2)$$
$$\cos\phi = (X + y\omega)/\sqrt{2\Phi}, \quad (3)$$
$$Q = 2k(Y + x\omega), \quad (4)$$
$$q = -(2X + y\omega)/(2k\omega), \quad (5)$$

where x, y, are Cartesian coordinates in Hill's rotating frame, X, Y, are their conjugate momenta, respectively, ω is the rotation rate of the system, and k is an scaling parameter of the transformation, whose value is chosen $k = \sqrt{3/4}$ for convenience, cf. [7].

Then, the Hill problem Hamiltonian is written

$$\mathcal{H} = \omega\Phi\left(1 - 3\xi^2 - \frac{\gamma}{\sqrt{\Delta^2 + \xi s + 2\eta c + \xi^2 + \eta^2}}\right), \quad (6)$$

in which the following abbreviations are used:

$$\Delta = (1 - k^2\sin^2\phi)^{1/2}, \quad (7)$$
$$\xi = Q/(2kB), \quad (8)$$
$$\eta = 2kq/a, \quad (9)$$
$$\gamma = \mu/(a\omega\Phi), \quad (10)$$
$$a = 2b, \quad (11)$$
$$B = b\omega, \quad (12)$$
$$b = (2\Phi/\omega)^{1/2}, \quad (13)$$
$$c = \cos\phi, \quad s = \sin\phi. \quad (14)$$

The flow stemming from Eq. (6) can be geometrically interpreted like a drifting ellipse of semi-axes b and a in the ratio 1:2, whose guiding center, of coordinates

$$x_{\mathrm{C}} = 2b\xi = Q/(k\omega), \qquad y_{\mathrm{C}} = a\eta = 2kq, \quad (15)$$

evolves with slow librational motion about the smaller primary, which is roughly approximated by harmonic oscillations

$$x_C = \frac{\Omega}{k\omega}M\sin(\Omega t + \psi), \quad y_C = 2kM\cos(\Omega t + \psi), \quad (16)$$

with libration frequency $\Omega \ll \omega$ given by

$$\Omega = \omega\sqrt{\gamma\big[K(k^2) - E(k^2)\big]/\pi}, \quad (17)$$

where $K(k^2)$ and $E(k^2)$ are the complete elliptic integral of first and second kind, respectively, of modulus k, and

$$M = \sqrt{{q_0}^2 + (Q_0/\Omega)^2}, \quad \tan\psi = Q_0/(\Omega q_0), \quad (18)$$

cf. [7]. The phase of the satellite ϕ evolves fast when compared to the long-period oscillations of the guiding center, a fact that permits to investigate the evolution of the orbits by the usual averaging of short-period effects [8, 9, 14].

To do that, Eq. (6) is expanded in the form of the perturbation Hamiltonian

$$\mathcal{H} = \sum_{i\geq 0}\frac{\epsilon^i}{i!}H_{i,0}(\phi, q, \Phi, Q), \quad (19)$$

in which ϵ is a formal small parameter, and the expansions are made under the following assumptions

$$\eta = \mathcal{O}(\epsilon), \qquad \xi = \mathcal{O}(\epsilon^2), \qquad \gamma = \mathcal{O}(\epsilon^4). \quad (20)$$

Then, we search for a canonical transformation

$$T : (\phi, q, \Phi, Q; \omega) \to (\phi', q', \Phi', Q'), \quad (21)$$

such that, in the new, prime variables, the Hamiltonian (19) can be written as $(T : \mathcal{H}) \equiv \mathcal{H}'$, where

$$\mathcal{H}' = \sum_{i=0}^{n}\frac{\epsilon^i}{i!}H_{0,i}(-, q', \Phi', Q') + \epsilon^{n+1}\mathcal{R}(\phi', q', \Phi', Q'). \quad (22)$$

After truncation of Eq. (22) to the order of ϵ^n, the new Hamiltonian does not depend on ϕ', and, in consequence, Φ' becomes a formal integral of the Hill problem in those regions of phase space in which the assumptions in Eq. (20) are valid.

The removal of short-period effects is made with the Lie transforms method [6]. It is summarized in solving the homological equation

$$H_{0,m} = \{H_{0,0}; W_m\} + \tilde{H}_{0,m}, \quad (23)$$

in which terms $H_{0,m}$ of the new Hamiltonian (22) are chosen at our convenience, terms $\tilde{H}_{0,m}$ are known from the original Hamiltonian (19) or from previous computations based on Deprit's triangle

$$H_{n,q+1} = H_{n+1,q} + \sum_{0\leq m\leq n}\binom{n}{m}\{H_{n-m,q}; W_{m+1}\} \quad (24)$$

and terms W_m, which comprise the generating function

$$W = \sum_{i\geq 0}\frac{\epsilon^i}{i!}W_{i+1,0}(\phi, q, \Phi, Q), \quad (25)$$

are solved at each step of the procedure by solving the partial differential equation (23), in which the symbol $\{\,;\}$ denotes the Poisson bracket operator.

In particular, terms $H_{0,m}$ are chosen by averaging short-period effects: $H_{0,m} = \langle\tilde{H}_{0,m}\rangle_\phi$, and, in view of $H_{0,0} = \omega\Phi$, terms W_m are solved by quadrature, viz.

$$W_m = (1/\omega)\int(\tilde{H}_{0,m} - H_{0,m})\,\mathrm{d}\phi + C_m. \quad (26)$$

where $C_m \equiv C_m(-, q, \Phi, Q)$ is an integration "constant". In order to guarantee that W_m is as much as possible free from long-period effects, it is customary to choose such C_m that $\langle W_m\rangle_\phi = 0$ [15, 16, 17, 18, 19].

Once the generating function is computed up to the desired truncation order, the series representing the equations of the transformation T:

$$\phi = \sum_{i\geq 0}\frac{\epsilon^i}{i!}\phi_{0,i}(\phi', q', \Phi', Q'),$$

and analogous series for the other variables, are directly constructed by standard application of Deprit's triangle (24) by simply noting that the original variables can be written in the form of Taylor series expansions in which the zeroth order term is the variable itself, and terms indexed $(i, 0)$ are null.

3. SHORT PERIOD AVERAGING

Application of the Lie transforms procedure to remove short-period effects from Eq. (19) is detailed below. For the sake of brevity in following expressions, we define the auxiliary quantities $\tilde{K} = K(k^2)/\pi$, $\tilde{E} = E(k^2)/\pi$, and the auxiliary functions

$$\begin{aligned} F^* &= 2\tilde{K}\,\phi - F(\phi\,|\,k^2) && (27)\\ E^* &= 2\tilde{E}\,\phi - E(\phi\,|\,k^2) && (28)\\ P^* &= 2\phi - \Pi(k^2;\phi\,|\,0) && (29) \end{aligned}$$

where $F(\phi\,|\,k^2)$ and $E(\phi\,|\,k^2)$ note the incomplete elliptic integral of first and second kind, respectively, and $\Pi(\alpha^2;\phi\,|\,k^2)$ is the elliptic integral of the third kind. Note that F^*, E^*, and P^*, are periodic function of ϕ with period π. Remark that the standard analogy between $\Pi(\alpha^2;\phi\,|\,0)$ and the arc-tangent function is not applied here because we are interested in evaluating ephemeris without constrain to the interval $\phi \in [0, \pi/2)$.

3.1. 1st to 4th orders

Since $H_{1,0} = H_{2,0} = H_{3,0} = 0$, we choose $H_{0,1} = H_{0,2} = H_{0,3} = 0$, and compute $W_1 = W_2 = W_3 = 0$, from which the periodic corrections vanish up to the third order ($\phi_{0,1} = \phi_{0,2} = \cdots = Q_{0,3} = 0$) and Deprit's triangle is filled with the terms $H_{1,1} = H_{1,2} = H_{2,1} = 0$ derived from Eq. (24).

The first non-vanishing term of Eq. (19) is $H_{4,0} = 4!(-B^2/2)(3\xi^2 - \gamma/\Delta)$. From Deprit's triangle, we compute $\tilde{H}_{0,4} = H_{4,0}$, and choose

$$H_{0,4} = \langle H_{4,0}\rangle_\phi = 4!(-\omega\Phi)\left(2\gamma\tilde{K} + 3\xi^2\right). \quad (30)$$

Next, we compute W_4 using Eq. (26), and, from it, using Eq. (24), $q_{0,4} = 0$, $Q_{0,4} = 0$,

$$\frac{\phi_{0,4}}{4!} = -\frac{1}{2}\gamma F^*, \qquad \frac{\Phi_{0,4}}{4!\Phi} = \gamma\left(\frac{1}{\Delta} - 2\tilde{K}\right). \quad (31)$$

3.2. 5th order

Using the assumptions in Eq. (20), terms $H_{i,0}$, $i > 4$, are obtained from the standard series expansion of the last term in the brackets of Eq. (6), which must be further multiplied by $\omega\Phi$. For $i = 5$, Deprit's recursion (24) gives $\tilde{H}_{5,0} = H_{5,0}$, and we choose $H_{0,5} = \langle\tilde{H}_{0,5}\rangle_\phi = 0$. Then, we compute W_5 from Eq. (26), and $q_{0,5} = 0$,

$$\frac{\phi_{0,5}}{5!} = -\gamma\frac{\eta}{\Delta}s, \quad \frac{\Phi_{0,5}}{5!\Phi} = -\gamma\frac{\eta}{\Delta^3}c, \quad \frac{Q_{0,5}}{5!B} = -\frac{k}{2}\frac{\gamma}{\Delta}s.$$

3.3. 6th order

Again, Eq. (24) leads to $\tilde{H}_{0,6} = H_{6,0}$. Hence,

$$H_{0,6} = \langle\tilde{H}_{0,6}\rangle_\phi = 6!(-\omega\Phi)\gamma\eta^2\left(\tilde{K} - \tilde{E}\right)/k^2. \quad (32)$$

It follows the standard computation of W_6 from Eq. (26) and

$$\begin{aligned} \frac{\phi_{0,6}}{6!} &= \gamma\Big\{\frac{c}{\Delta}\Big[\Big(\frac{1}{\Delta^2}+1\Big)\eta^2k^2s + 2\xi\Big] + \eta^2[E^* - F^*]\Big\}\\ \frac{q_{0,6}}{6!b} &= \frac{\gamma}{2k}\frac{1}{\Delta}\cos\phi\\ \frac{\Phi_{0,6}}{6!\Phi} &= \gamma\Big\{\frac{4}{3}\eta^2(\tilde{E} - \tilde{K}) + \frac{1}{2\Delta^3}\Big[\Big(3 - \frac{1}{\Delta^2}\Big)\eta^2 - \xi s\Big]\Big\}\\ \frac{Q_{0,6}}{6!B} &= \frac{\gamma}{2k}\eta\Big[\frac{k^2}{\Delta}\Big(\frac{1}{\Delta^2}+1\Big)cs + E^* - F^*\Big] \end{aligned}$$

3.4. 7th order

Once more, Deprit's recursion yields $\tilde{H}_{0,7} = H_{7,0}$, and we choose $H_{0,7} = \langle\tilde{H}_{0,7}\rangle_\phi = 0$. It follows the standard computation of W_7 from the quadrature Eq. (26), and

$$\begin{aligned} \frac{\phi_{0,7}}{7!} &= \gamma\eta\Big[\Big(\frac{1}{\Delta^3} - 8\tilde{E}\Big)\xi + \frac{1}{9}\frac{s}{\Delta}\Big(\frac{3}{\Delta^4} - \frac{7}{\Delta^2} - 14\Big)\eta^2\Big]\\ \frac{q_{0,7}}{7!b} &= -\frac{1}{6k}\gamma\eta\Big(\frac{1}{\Delta^3} - 8\tilde{E}\Big)\\ \frac{\Phi_{0,7}}{7!\Phi} &= \frac{\gamma\eta}{2\Delta^5}\Big[\frac{1}{3}\Big(\frac{5}{\Delta^2} - 11\Big)\eta^2 + 3\xi s\Big]c\\ \frac{Q_{0,7}}{7!B} &= \frac{k}{3}\gamma\Big[\Big(\frac{1}{\Delta^3} - 8\tilde{E}\Big)\xi + \frac{1}{4}\frac{s}{\Delta}\Big(\frac{3}{\Delta^4} - \frac{7}{\Delta^2} - 14\Big)\eta^2\Big] \end{aligned}$$

3.5. 8th order

At this step, the coupling between perturbations of different orders take place for the first time. Now, from Eq. (24) we find $\tilde{H}_{0,8} = 35\{H_{0,4}; W_4\} + 35\{H_{4,0}; W_4\} + H_{8,0}$. We choose $H_{0,8} = \langle\tilde{H}_{0,8}\rangle_\phi$, namely

$$H_{0,8} = 8!\omega\Phi\gamma\Big[\gamma(1 - 2\tilde{K}^2) + \frac{14\tilde{E} - 11\tilde{K}}{9}\eta^4 + \frac{\tilde{K} - 4\tilde{E}}{k^2}\xi^2\Big] \quad (33)$$

and compute W_8 from Eq. (26), from which we obtain

$$\begin{aligned} \frac{\phi_{0,8}}{8!} &= \gamma\Big\{\gamma\Big[P^* - \Big(2k^2\tilde{K} + \frac{1}{4\Delta}\Big)F^*\Big]\\ &\quad + \frac{5}{36}(14E^* - 11F^*)\eta^4 + (F^* - 4E^*)\xi^2\\ &\quad + \frac{1}{\Delta}\Big[\frac{2}{3}\Big(8 + \frac{1}{\Delta^2} - \frac{3}{\Delta^4}\Big)\eta^2\xi - \Big(3 + \frac{k^2}{\Delta^2}\Big)\xi^2 s\\ &\quad + \frac{5}{48}\Big(14 + \frac{11}{\Delta^2} + \frac{8}{\Delta^4} - \frac{5}{\Delta^6}\Big)\eta^4 s\Big]c\Big\},\\ \frac{q_{0,8}}{8!b} &= \frac{\gamma}{k}\Big\{\frac{1}{3}(4E^* - F^*)\xi + \frac{c}{4\Delta}\times\\ &\quad \times\Big[\Big(\frac{1}{\Delta^4} - \frac{1}{3\Delta^2} - \frac{8}{3}\Big)\eta^2 + \Big(\frac{1}{\Delta^2} + 4\Big)\xi s\Big]\Big\} \end{aligned}$$

$$
\begin{aligned}
\frac{\Phi_{0,8}}{8!\Phi} &= \gamma\Big\{\gamma\Big[1-\frac{1}{2\Delta^2}+\tilde{K}\Big(\frac{1}{\Delta}-2\tilde{K}\Big)-\frac{k^2}{2\Delta^3}F^*sc\Big] \\
&\quad +\xi^2\Big[\frac{1}{\Delta^3}\Big(\frac{1}{2\Delta^2}-1\Big)+\frac{\tilde{K}-4\tilde{E}}{k^2}\Big] \\
&\quad +\frac{\eta^4}{9}\Big[\frac{1}{8\Delta^5}\Big(\frac{35}{\Delta^4}-\frac{190}{\Delta^2}+227\Big)+14\tilde{E}-11\tilde{K}\Big] \\
&\quad +\frac{\eta^2\xi}{4\Delta^5}\Big(\frac{5}{\Delta^2}-17\Big)s\Big\} \\
\frac{Q_{0,8}}{8!B} &= \frac{k}{9}\gamma\Big\{(14E^*-11F^*)\eta^2+\frac{3}{\Delta}\Big[\xi\Big(8+\frac{1}{\Delta^2}-\frac{3}{\Delta^4}\Big) \\
&\quad +\eta^2\Big(\frac{7}{2}+\frac{11}{4\Delta^2}+\frac{2}{\Delta^4}-\frac{5}{4\Delta^6}\Big)s\Big]c\Big\}\eta
\end{aligned}
$$

3.6. 9th order

Now, the known terms from previous computations yield $\tilde{H}_{0,9}=56\{H_{0,4};W_5\}+70\{H_{4,0};W_5\}+56\{H_{5,0};W_4\}+H_{9,0}$, and the average over ϕ gives $H_{0,9}=\langle\tilde{H}_{0,9}\rangle_\phi=0$. The ninth term of the generating function is computed as usual, and

$$
\begin{aligned}
\frac{\phi_{0,9}}{9!} &= \gamma\Big\{\frac{\gamma\eta}{\Delta}\Big[s\Big(\frac{11}{8\Delta}-4\tilde{K}\Big)+\frac{cF^*}{2\Delta^2}\Big] \\
&\quad +\xi\Big[k\log 8(\Delta+kc)^2+\frac{5\eta^3}{18\Delta^5}\Big(19-\frac{5}{\Delta^2}\Big)\Big] \\
&\quad +\frac{5}{4}\gamma\eta k\log\Big(\frac{1+ks}{1-ks}\Big)+\frac{\eta\xi^2}{3\Delta}\Big(8+\frac{4}{\Delta^2}-\frac{3}{\Delta^4}\Big)s \\
&\quad -\frac{\eta^5 s}{4\Delta}\Big(\frac{7}{6\Delta^8}-\frac{17}{3\Delta^6}+\frac{33}{10\Delta^4}+\frac{22}{5\Delta^2}+\frac{44}{5}\Big)\Big\}, \\
\frac{q_{0,9}}{9!b} &= \frac{\gamma}{k}\frac{1}{4}\Big\{\frac{\eta^3}{9\Delta^5}\Big(\frac{5}{\Delta^2}-19\Big)-k\log 8(\Delta+kc)^2 \\
&\quad +\frac{\eta\xi}{k^2\Delta}\Big(\frac{k^2}{\Delta^4}-\frac{1}{\Delta^2}-2\Big)s\Big\} \\
\frac{\Phi_{0,9}}{9!\Phi} &= \gamma\Big\{\frac{k^2\xi}{\Delta}\Big[\frac{5\eta^3}{9\Delta^6}\Big(19-\frac{7}{\Delta^2}\Big)c-2\Big]s+\frac{c\eta}{\Delta^5} \\
&\quad \times\Big[\Big(4-\frac{5}{2\Delta^2}\Big)\xi^2-\frac{\eta^4}{8\Delta^2}\Big(\frac{7}{\Delta^4}-\frac{98}{3\Delta^2}+\frac{101}{3}\Big)\Big] \\
&\quad +\frac{\gamma\eta}{2\Delta^2}\Big[\Big(2-\frac{4\tilde{K}}{\Delta}+\frac{1}{\Delta^2}\Big)c-\frac{F^*}{\Delta}\Big(\frac{k^2}{\Delta^2}-2\Big)s\Big]\Big\} \\
\frac{Q_{0,9}}{9!B} &= \frac{k}{9}\gamma\Big\{\frac{\gamma}{2\Delta}\Big[s\Big(\frac{k^2}{\Delta}-2\tilde{K}\Big)+\frac{cF^*}{2\Delta^2}\Big] \\
&\quad +\frac{\eta^2\xi}{6\Delta^5}\Big(19-\frac{5}{\Delta^2}\Big)+\frac{\xi^2 s}{3\Delta}\Big(2-\frac{k^2}{\Delta^4}+\frac{1}{\Delta^2}\Big) \\
&\quad +\frac{1}{4}\gamma k\log\Big(\frac{1+ks}{1-ks}\Big) \\
&\quad -\frac{\eta^4 s}{2\Delta}\Big(\frac{35}{72\Delta^8}-\frac{85}{36\Delta^6}+\frac{11}{8\Delta^4}+\frac{11}{6\Delta^2}+\frac{11}{3}\Big)\Big\}
\end{aligned}
$$

3.7. 10 to 13th orders

Now, the known terms from previous computations yield $\tilde{H}_{0,10}=84\{H_{0,4};W_6\}+126\{H_{0,6};W_4\}+126\{H_{4,0};W_6\}+126\{H_{5,0};W_5\}+84\{H_{6,0};W_4\}+H_{10,0}$. Then, we choose $H_{0,10}=\langle\tilde{H}_{0,10}\rangle_\phi$, to get

$$
\begin{aligned}
H_{0,10} &= 10!2\omega\Phi\gamma\eta^2\Big\{\gamma\Big[1+\frac{8}{3}(\tilde{E}-\tilde{K})\tilde{K}\Big] \\
&\quad +\frac{71\tilde{E}-50\tilde{K}}{81}\eta^4+\frac{8\tilde{K}-20\tilde{E}}{3}\xi^2\Big\}
\end{aligned}
$$

At this stage we found integrals that we failed to solve analytically in the solution of Eq. (26), so the transformation equations of the short-period elimination is only computed up to the ninth order. Nevertheless, because the peculiar arrangement we chose for the Hamiltonian, Hamiltonian terms in prime variables, which include the coupling effects of the perturbation, are successfully computed up to the order 13th. Corresponding terms are listed below.

$$
\begin{aligned}
H_{0,11} &= 11!(-\omega\Phi)\gamma^2/2 \\
H_{0,12} &= 12!\omega\Phi\gamma\Big\{5\gamma^2\Big(\frac{4}{15}+\tilde{K}-\tilde{E}-\tilde{K}^3\Big) \\
&\quad +\frac{\gamma}{3}\Big[(9-8\tilde{E}^2+44\tilde{E}\tilde{K}-30\tilde{K}^2)\eta^4 \\
&\quad +8(2\tilde{K}^2-8\tilde{E}\tilde{K}+3)\Big]+\frac{\eta^8}{324}(644\tilde{E}-425\tilde{K}) \\
&\quad -\frac{10}{27}\eta^4\xi^2(74\tilde{E}-35\tilde{K})+\frac{4}{9}\xi^4(\tilde{K}-16\tilde{E})\Big\} \\
H_{0,13} &= 13!(-1)\omega\Phi\gamma\eta\Big\{\frac{16}{13}\gamma^2+\frac{\tilde{K}+26\tilde{E}}{k^2}\xi^2\eta \\
&\quad +\Big[\frac{5}{6}-\frac{32}{9}\tilde{E}(\tilde{E}-\tilde{K})\Big]\gamma\eta\Big\}
\end{aligned}
$$

Thus, the long-term Hamiltonian, after truncation to higher order terms, is

$$
\mathcal{H}'=\sum_{i=0}^{13}\frac{\epsilon^i}{i!}H_{0,i}(-,q',\Phi',Q'), \tag{34}
$$

which is of one degree of freedom in (q',Q') and depends on the dynamical parameter Φ'. The evolution of the orbit can then be obtained from the numerical integration of the Hamilton equations derived from Eq. (34), which progress with very long steps because it is free from short-period effects [14]. However, a high order analytical solution is also feasible, that provides results of comparable accuracy to the numerical integration of the evolution equations [7], which is described in Section 4.

4. LIDSTEDT SERIES APPROACH

The Hamiltonian flow stemming from Eq. (34) decouples into a reduced system

$$
\frac{\mathrm{d}q'}{\mathrm{d}t}=\frac{\partial\mathcal{H}'(q',Q';\Phi')}{\partial Q'},\qquad \frac{\mathrm{d}Q'}{\mathrm{d}t}=-\frac{\partial\mathcal{H}'(q',Q';\Phi')}{\partial q'}. \tag{35}
$$

and a quadrature

$$\phi = \phi_0 + \int \frac{\partial \mathcal{H}'(q'(t), Q'(t); \Phi')}{\partial \Phi'} \, \mathrm{d}t. \tag{36}$$

The latter is integrated after the solution $q' = q'(t, q'_0, Q'_0; \Phi')$, $Q' = Q'(t, q'_0, Q'_0; \Phi')$ of the reduced system (35) is known.

However, while the long-term Hamiltonian is integrable, time explicit solutions have been obtained only for the lower truncations. Alternatively, a Lindsted series solution to the reduced system (35) has been computed in [5], but limited to the 1st order of the Lindstedt approach. An extension to the second order of the Lindstedt series procedure, which besides provides the evolution of ϕ' given by Eq. (36), has been used in [7] without further description. We provide here full details on the Lindstedt solution for the DROs long-term evolution.

The basics of the Lindstedt series approach are as follows (see [20], for instance). First, we change the independent variable

$$\tau = nt. \tag{37}$$

The effect of this change is a simple scaling of Eqs. (35) and (36), which now read

$$n\frac{\mathrm{d}q'}{\mathrm{d}\tau} = \frac{\partial \mathcal{H}'(q', Q'; \Phi')}{\partial Q'}, \qquad n\frac{\mathrm{d}Q'}{\mathrm{d}\tau} = -\frac{\partial \mathcal{H}'(q', Q'; \Phi')}{\partial q'}. \tag{38}$$

and

$$\phi' = \phi'_0 + \frac{1}{n} \int \frac{\partial \mathcal{H}'(q'(\tau), Q'(\tau); \Phi')}{\partial \Phi'} \, \mathrm{d}\tau. \tag{39}$$

Next, the series

$$n = 1 + \sum_{i\geq 1} \varepsilon^i n_i, \quad q' = \sum_{i\geq 0} \varepsilon^i q^{(i)}(\tau), \quad Q' = \sum_{i\geq 0} \varepsilon^i Q^{(i)}(\tau), \tag{40}$$

where ε is just formal, are replaced into Eqs. (38) and (39). Then, after equating the coefficients of equal powers of ε, we obtain a chain of differential systems that can be solved sequentially as a function of τ and the initial conditions q'_0, Q'_0. Finally, for each system i, we must determine the coefficient n_i so that the solution $(q^{(i)}(\tau), Q^{(i)}(\tau))$ be periodic.

At each step of the procedure, we also compute the corresponding term $\phi_i(\tau)$ from Eq. (39), so that

$$\phi = \phi_0 + \frac{1}{n} \sum_{i\geq 0} \varepsilon^i \phi^{(i)}(\tau). \tag{41}$$

Up to the second order of ε we get

$$q' = q^{(0)}(\tau) + q^{(1)}(\tau) + q^{(2)}(\tau), \tag{42}$$

$$Q' = Q^{(0)}(\tau) + Q^{(1)}(\tau) + Q^{(2)}(\tau), \tag{43}$$

$$\phi' = \phi'_0 + \frac{1}{n}\left[\phi^{(0)}(\tau) + \phi^{(1)}(\tau) + \phi^{(2)}(\tau)\right], \tag{44}$$

The later is better written in the form

$$\phi' = \phi'_0 + \frac{1}{n}\left\{\omega\left(1 + \frac{\Omega^2}{\omega^2} d\right)\tau + \frac{\Omega}{\omega}\left[p(\tau) - p(0)\right]\right\}, \tag{45}$$

where

$$d = d^{(0)} + d^{(1)} + d^{(2)}, \tag{46}$$

$$p = p^{(0)}(\tau) + p^{(1)}(\tau) + p^{(2)}(\tau). \tag{47}$$

Specifically, the Lindstedt series we obtained are conveniently arranged in the form of the following summations:

$$n = \sum_{m=0}^{2}\sum_{j=0}^{m}\sum_{k=0}^{m-j}\left(\frac{\Omega}{\omega}\right)^{2m-2j-2k} \times \left(\frac{Q'_0/\Omega}{b}\right)^{2j}\left(\frac{q'_0}{b}\right)^{2k} n_{m,j,k} \tag{48}$$

$$q' = \sum_{m=0}^{2}\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{m-j}\left(\frac{\Omega}{\omega}\right)^{2m-2j-2k}\left(\frac{Q'_0/\Omega}{b}\right)^{2j} \times \left(\frac{q'_0}{b}\right)^{2k}\Big[c_{m,i,j,k}q'_0\cos(2i+1)\Omega\tau + s_{m,i,j,k}(Q'_0/\Omega)\sin(2i+1)\Omega\tau\Big] \tag{49}$$

$$Q' = \sum_{m=0}^{2}\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{m-j}\left(\frac{\Omega}{\omega}\right)^{2m-2j-2k}\left(\frac{Q'_0/\Omega}{b}\right)^{2j} \times \left(\frac{q'_0}{b}\right)^{2k}\Big[C_{m,i,j,k}Q'_0\cos(2i+1)\Omega\tau + S_{m,i,j,k}(q'_0\Omega)\sin(2i+1)\Omega\tau\Big] \tag{50}$$

$$\begin{aligned} p = {} & \frac{(64/13)k}{(\mathrm{K}-\mathrm{E})^3}\frac{\Omega^4}{\omega^4}\left(\frac{q'_0}{b}\sin\Omega\tau + \frac{Q'_0/\Omega}{b}\cos\Omega\tau\right) \\ & + \frac{q'_0}{b}\frac{Q'_0/\Omega}{b}\sum_{m=0}^{2}\sum_{i=1}^{m+1}\sum_{j=0}^{m}\sum_{k=0}^{m-j}\left(\frac{\Omega}{\omega}\right)^{2m-2j-2k} \times \left(\frac{Q'_0/\Omega}{b}\right)^{2j}\left(\frac{q'_0}{b}\right)^{2k}\kappa_{m,i,j,k}\cos 2i\Omega\tau \\ & + \sum_{m=0}^{2}\sum_{i=1}^{m+1}\sum_{j=0}^{2m+1}\sum_{k=0}^{m+1-j}\left(\frac{\Omega}{\omega}\right)^{2m+2-2j-2k} \times \left(\frac{Q'_0/\Omega}{b}\right)^{2j}\left(\frac{q'_0}{b}\right)^{2k}\sigma_{m,i,j,k}\sin 2i\Omega\tau \end{aligned} \tag{51}$$

$$d = \frac{\mathrm{K}}{\mathrm{K}-\mathrm{E}} + \sum_{m=0}^{2}\sum_{j=0}^{m+1}\sum_{k=0}^{m+1-j}\left(\frac{\Omega}{\omega}\right)^{2+2m-2j-2k} \times \left(\frac{Q'_0/\Omega}{b}\right)^{2j}\left(\frac{q'_0}{b}\right)^{2k} d_{m,j,k} \tag{52}$$

where the exact numeric coefficients $n_{m,j,k}$, $c_{m,i,j,k}$, $s_{m,i,j,k}$, $C_{m,i,j,k}$, $S_{m,i,j,k}$, $\kappa_{m,i,j,k}$, $\sigma_{m,i,j,k}$, and $d_{m,j,k}$, are given in Tables 1–8 of the Appendix.

Orbital (T_O) and libration periods (T_L) are then defined

$$T_O = \frac{2\pi}{\omega\left[1 + (\Omega/\omega)^2 d\right]}, \qquad T_L = \frac{2\pi}{\Omega n}, \tag{53}$$

and a periodic solution, on average, will happen when both periods are commensurable.

5. NUMERICAL EXPERIMENTS

We only need two basic parameters to define a distant retrograde orbit: the size of the drifting ellipse and the minimum distance of it to the smaller primary in the y axis direction, cf. [14].

The first is given by any of the semi-axes, from which Φ is computed from Eq. (13), and Ω from Eqs. (17) and (10). The second is given by $\rho = a - 2kM$, from which $M = (a-\rho)/(2k)$. On the other hand, if we chose $q_0 = 0$, then $M = Q_0/\Omega$ from Eq. (18). Therefore $Q_0 = \Omega(a-\rho)/(2k)$. Another straightforward possibility is to choose $Q_0 = 0$, hence $M = q_0$ and, therefore, $q_0 = (a-\rho)/(2k)$. We are free to choose any value of ϕ to complete an initial set of initial conditions of a DRO with the desired characteristics in the averaged, prime space.

As a first example, we search for the typical 1:1 DRO. We choose $a = \rho = 10$ in Hill units ($\mu = \omega = 1$), and compute $\Phi' = 12.5$. Despite $\Omega = 0.0490672$ for this value of a, and hence our solution predicts a libration period about 20 times higher than the orbital one, this would not fit to the 1:1 resonance, so we impose $T_L = T_O = 6.24852$, as obtained from the left equation of Eq. (53). We fix $q'_0 = 0$ and compute $Q'_0 = 0$, which immediately shows that terms in the square brackets of Eqs. (49) and (50) always vanish, resulting in a fixed guiding center at the origin, on average. The orbit libration still exists, as checked from the numerical propagation of corresponding initial conditions in the original (non-averaged) Hill problem equations. Nevertheless, the errors of the analytical theory are very small, and almost negligible when the short-period corrections are incorporated into the analytical solution.

Indeed, as shown in Fig. 1, the amplitude of the errors of the mean elements propagation remains of just a few thousandths, and practically vanish when the short-period corrections are applied. Errors grow higher for the phase of the orbiter, and a secular trend due to the early truncation of the theory is clearly observed in the time history of the errors of ϕ.

If desired, the initial conditions provided by the analytical solution can be improved with differential corrections to get an exact periodic orbit in the 1:1 resonance. Using the algorithm of [21], we find that just 3 iterations are enough to obtain the initial conditions of a true periodic, stable, DRO of the Hill problem with initial conditions $x = Y = 0$, $y = 9.783444749944893$, and $X = -4.847560254601411$, that, after propagated for an improved period $T_O = 6.247084797518564$ yield periodicity errors of the order of 10^{-12} for x and Y, and $\mathcal{O}(10^{-10})$ for y and X.

Another example is presented for the case of larger libration, which is definitely much more challenging. Now we

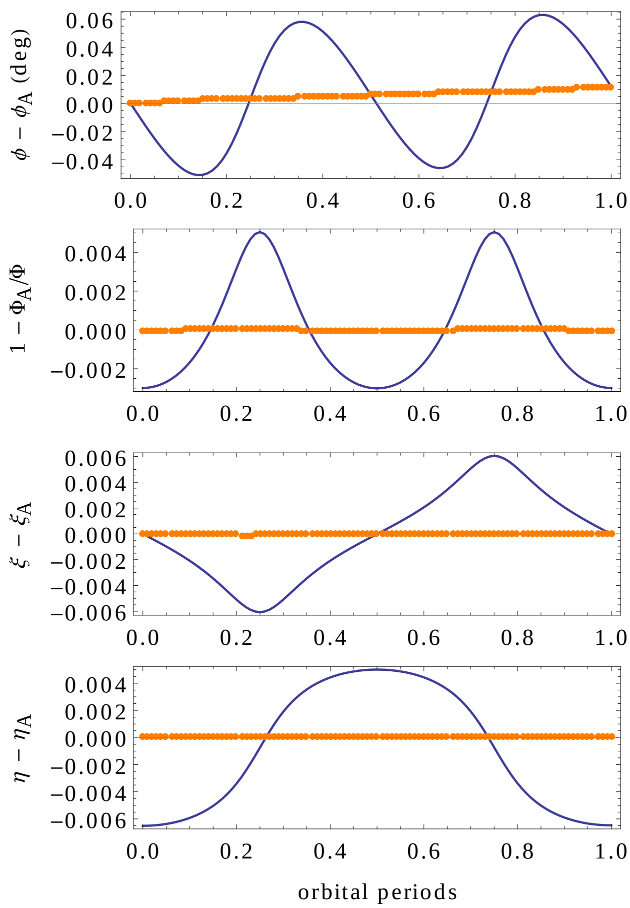

Fig. 1. Errors of the orbit with design parameters $a = \rho = 10$, with respect to the mean elements prediction alone (solid line), and with short-period corrections (light dots).

choose a DRO with $a = 10$, $\rho = 5$ in Hill units ($\mu = \omega = 1$). Next, we compute $\Phi' = 12.5$ and $\Omega = 0.0490672$, about 20 times smaller than ω. Then, for $q'_0 = 0$ we compute $Q'_0 = 0.141645$ and choose $\phi'_0 = 0$. The analytical propagation of these initial conditions for a libration period shows that ϕ' grows with an almost linear rate which is just slightly higher than 2π times orbital period, Φ', of course, remains constant, and the evolution of the non-dimensional version of Q and q, given in Eqs. (8) and (9), respectively, is depicted in Fig. 2.

If we now compute initial conditions in the original, non averaged space and compute the corresponding orbit by numerical propagating the equations of motion of the Hill problem, we find that the real orbit and the analytical prediction match quite well, as depicted in the left plot of Fig. 3 where the mean orbit is represented with light dots and the real orbit with a solid line. On the other hand, the predicted behavior of the guiding center indeed averages the trajectory derived from the real trajectory, as shown in the right plot of Fig. 3,

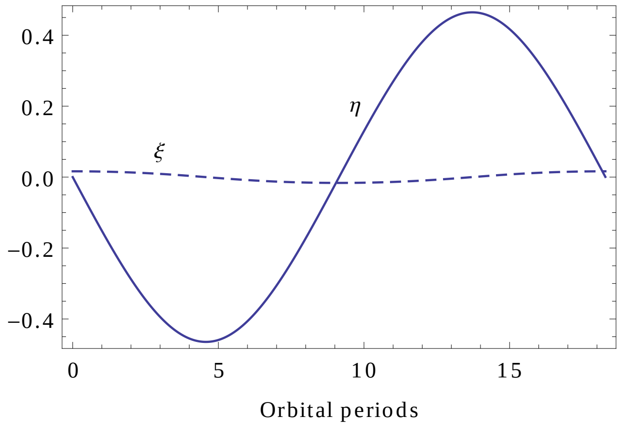

Fig. 2. Non-dimensional equinoctial variables ξ (dashed line) and η (solid line) along one libration period of the orbit with design parameters $a = 10$, $\rho = 5$.

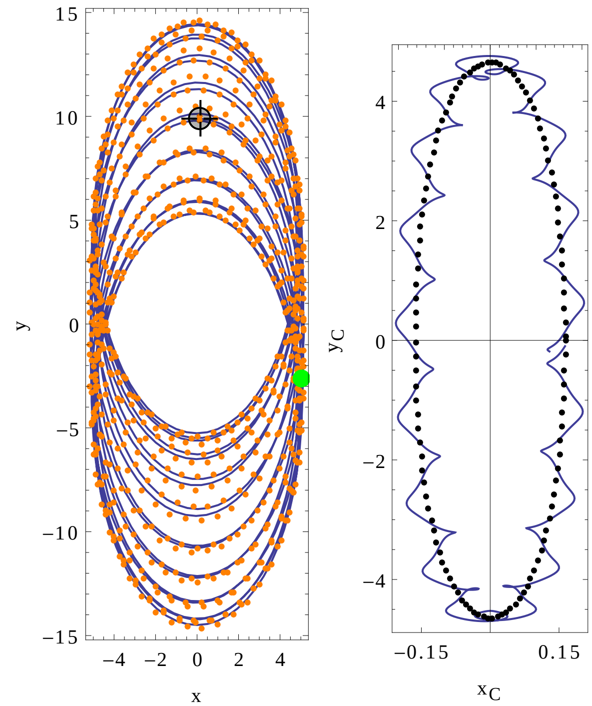

Fig. 3. Orbit with design parameters $a = 10$, $\rho = 5$, evaluated along one libration period ~ 18.3 orbital periods. Left: Mean orbit prediction (light dots) superimposed to the numerical propagation (solid line); the bright dot at $\sim (5, -2.6)$ marks the last point for the predicted libration period. Right: Mean evolution of the guiding center (black dots) superimposed to the true trajectory (solid line).

where the analytical prediction is represented with dots.

Things fit even much better when short periodic effects are added to the mean elements prediction. Indeed, as shown in Fig. 4, short-period effects are the more important source of errors of the mean analytical solution (solid lines), whereas osculating predictions flatten the errors and (light dots) clearly disclose the secular and long-period errors that result from the early truncation of the perturbation approach.

The lack of commensurability between the orbital and libration periods $T_L/T_O = 18.3$ shows that the orbit with design parameters $a = 10$, $\rho = 5$ is not periodic. However, as shown in the left plot of Fig. 3, it is close to periodicity. Then, the initial conditions are amenable of improving by differential corrections to give a true periodic orbit. To do that, we use the algorithm described in [21], which need an initial estimation of the period. The initial conditions and period predicted by the analytical theory produce final conditions far away from the initial ones, as demonstrated by the initial point of coordinates $\sim (0.16, 9.9)$ and the final ones $\sim (5, -2.6)$, highlighted with a bright, large dot in the left plot of Fig. 3. Still, more favorable initial conditions and period are easily computed by linear interpolation using, for instance, the secant method. If we do that without modifying the minimum distance ρ, we obtain an exact commensurability $T_L/T_O = 18$ for the slightly modified semi-major axis $a = 9.87661$. Direct propagation of the improved initial conditions in the Hill (non-averaged) problem equations yield now an almost periodic orbit where initial and final points almost superimpose both for the coordinates and conjugate momenta, with absolute errors $\delta x \approx \delta Y = 0.01$, $\delta y \approx \Delta X = 0.17$.

Application of differential corrections to the improved initial conditions and period with the algorithm of [21], needs just 4 iterations to converge to a true periodic, stable, DRO of the Hill problem with initial conditions $x = 5.073172530052394$, $y = 0$, $X = 0.1353185618586326$, $Y = -5.014034636487915$ and period 112.3809318954195, producing periodicity errors $\mathcal{O}(10^{-14})$ for δx and δY, and $\mathcal{O}(10^{-11})$ for δy and ΔX.

6. CONCLUSIONS

A detailed description of an algorithm for computing distant retrograde orbit is provided. It relies on time explicit solutions of the Hill problem computed by perturbation methods, and approximates remarkably well the dynamics of this class of quasi-satellite orbits. The algorithm shows the existence of two design parameters, which are related to the size of the orbit and the minimum libration distance to the co-orbital primary. The proper choice of these parameter provides the initial conditions of an orbit with the characteristics determined by the user that is periodic on average. These initial conditions are amenable to the standard improvement with differential corrections, so that a true periodic orbit of the Hill problem with the desired attributes is easily obtained.

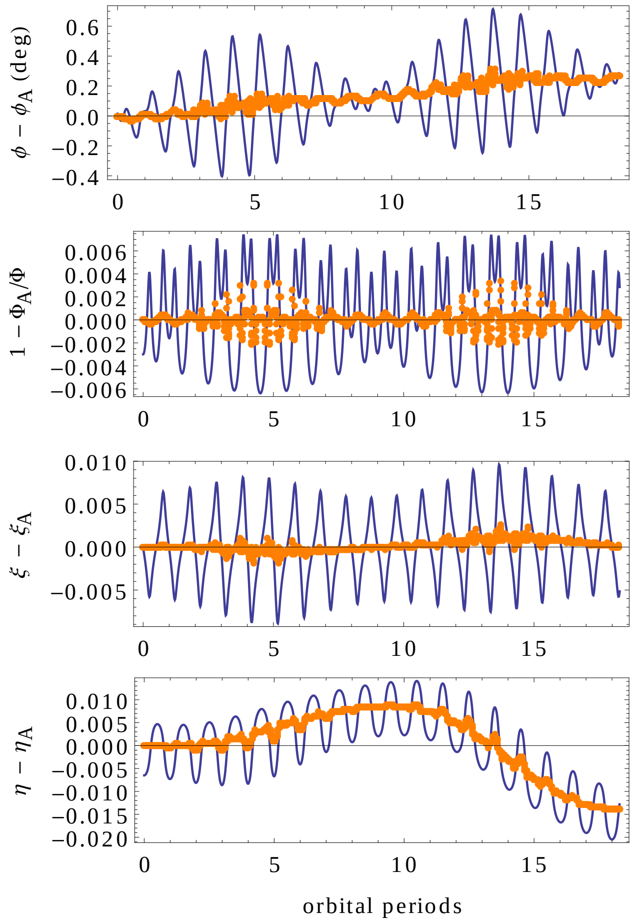

Fig. 4. Errors of the orbit with design parameters $a = 10$, $\rho = 5$, with respect to the mean elements prediction alone (solid line), and with short-period corrections (light dots).

Table 1. Coefficients $n_{m,j,k}$

$$n_{0,0,0} = 1$$
$$n_{1,0,0} = -\frac{-64EK-256K^2+320E^2+63}{144(E-K)^2}$$
$$n_{1,0,1} = n_{1,1,0} = \frac{3(11K-14E)}{64(K-E)}$$
$$n_{2,0,0} = -\frac{2(4E-K)^2}{81(E-K)^2}$$
$$n_{2,0,1} = -\frac{-370EK+35K^2+344E^2}{96(E-K)^2}$$
$$n_{2,0,2} = \frac{-12892EK+5459K^2+7244E^2}{16384(E-K)^2}$$
$$n_{2,1,0} = -\frac{-162EK+19K^2+152E^2}{32(E-K)^2}$$
$$n_{2,1,1} = 2n_{2,2,0} = -\frac{-1892EK+349K^2+2164E^2}{8192(E-K)^2}$$

Table 2. Coefficients $c_{m,i,j,k}$

$$c_{0,0,0,0} = 1, \qquad c_{1,0,0,0} = c_{1,1,0,0} = 0$$
$$c_{2,0,0,0} = c_{2,1,0,0} = c_{2,2,0,0} = c_{2,2,0,1} = c_{2,2,1,0} = 0$$
$$c_{1,0,1,0} = -c_{1,1,1,0} = \frac{3}{4}, \qquad c_{1,1,0,1} = -c_{1,0,0,1} = \frac{1}{4}$$
$$c_{2,0,0,1} = -c_{2,1,0,1} = \frac{12E-K}{16(E-K)}$$
$$c_{2,0,0,2} = -\frac{-9724EK+4451K^2+4652E^2}{196608(E-K)^2}$$
$$c_{2,0,1,0} = -\frac{-526EK+47K^2+488E^2}{192(E-K)^2}$$
$$c_{2,0,1,1} = -\frac{5(-7964EK+2971K^2+5452E^2)}{98304(E-K)^2}$$
$$c_{2,0,2,0} = -\frac{-26972EK+9019K^2+21004E^2}{196608(E-K)^2}$$
$$c_{2,1,0,2} = \frac{-286EK+137K^2+122E^2}{8192(E-K)^2}$$
$$c_{2,1,1,0} = \frac{-526EK+47K^2+488E^2}{192(E-K)^2}$$
$$c_{2,1,1,1} = \frac{5(-1804EK+689K^2+1196E^2)}{16384(E-K)^2}$$
$$c_{2,1,2,0} = \frac{3(-352EK+89K^2+344E^2)}{16384(E-K)^2}$$
$$c_{2,2,0,2} = \frac{-2860EK+1163K^2+1724E^2}{196608(E-K)^2}$$
$$c_{2,2,1,1} = -\frac{5(-2860EK+1163K^2+1724E^2)}{98304(E-K)^2}$$
$$c_{2,2,2,0} = \frac{5(-2860EK+1163K^2+1724E^2)}{196608(E-K)^2}$$

7. APPENDIX: EXACT COEFFICIENTS OF THE LIDSTEDT SERIES SOLUTION

Following tables provide the necessary coefficients of the Lindstedt series solution in Eqs. (48)–(52). The abbreviations $E \equiv E(k^2)$, $F \equiv F(k^2)$ are used for convenience.

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Table 3. Coefficients $s_{m,i,j,k}$

$$s_{0,0,0,0} = -1$$
$$s_{1,0,0,0} = \frac{128(4E-K)}{9(14E-11K)} - \frac{32EK-96K^2+64E^2+21}{48(E-K)^2}$$
$$s_{1,0,0,1} = \frac{21}{4}, \quad s_{1,0,1,0} = \frac{9}{4}, \quad s_{1,1,0,1} = -\frac{3}{4}, \quad s_{1,1,1,0} = \frac{1}{4}$$
$$s_{1,1,0,0} = s_{2,1,0,0} = s_{2,2,0,0} = s_{2,2,0,1} = s_{2,2,1,0} = 0$$
$$s_{2,0,0,0} = -n_{2,0,0}$$
$$s_{2,0,0,1} = \frac{-530EK-5K^2+472E^2}{384(E-K)^2}$$
$$s_{2,0,0,2} = \frac{11396EK+6563K^2-34132E^2}{196608(E-K)^2}$$
$$s_{2,0,1,0} = -\frac{-278EK+41K^2+264E^2}{128(E-K)^2}$$
$$s_{2,0,1,1} = -\frac{7(-17644EK+6143K^2+13148E^2)}{98304(E-K)^2}$$
$$s_{2,0,2,0} = -\frac{-95524EK+34373K^2+68468E^2}{196608(E-K)^2}$$
$$s_{2,1,0,1} = \frac{-994EK+83K^2+920E^2}{384(E-K)^2}$$
$$s_{2,1,0,2} = \frac{3(-968EK+331K^2+736E^2)}{16384(E-K)^2}$$
$$s_{2,1,1,0} = \frac{1}{4}n_{2,0,1}$$
$$s_{2,1,1,1} = -\frac{-5764EK+2363K^2+3428E^2}{16384(E-K)^2}$$
$$s_{2,1,2,0} = -\frac{-319EK+113K^2+233E^2}{4096(E-K)^2}$$
$$s_{2,2,0,2} = -5c_{2,2,0,2}$$
$$s_{2,2,1,1} = 10c_{2,2,0,2}, \quad s_{2,2,2,0} = -c_{2,2,0,2}$$

Table 4. Coefficients $C_{m,i,j,k}$

$$C_{0,0,0,0} = 1, \quad C_{1,0,0,0} = C_{1,1,0,0} = C_{2,0,0,0} = 0$$
$$C_{2,1,0,0} = C_{2,2,0,0} = C_{2,2,0,1} = C_{2,2,1,0} = 0$$
$$C_{1,1,0,1} = -C_{1,0,0,1} = \frac{9}{4}, \quad C_{1,0,1,0} = -C_{1,1,1,0} = \frac{3}{4}$$
$$C_{2,0,0,1} = -C_{2,1,0,1} = -3c_{2,0,0,1}$$
$$C_{2,0,0,2} = -\frac{-66748EK+32531K^2+27116E^2}{196608(E-K)^2}$$
$$C_{2,0,1,0} = -C_{2,1,1,0} = \frac{-98EK+K^2+88E^2}{192(E-K)^2}$$
$$C_{2,0,1,1} = -\frac{-65516EK+26527K^2+39772E^2}{98304(E-K)^2}$$
$$C_{2,0,2,0} = c_{2,0,2,0}$$
$$C_{2,1,0,2} = \frac{9(11EK+8K^2-37E^2)}{4096(E-K)^2}$$
$$C_{2,1,1,1} = \frac{3(-7612EK+3089K^2+4604E^2)}{16384(E-K)^2}$$
$$C_{2,1,2,0} = c_{2,1,2,0}, \quad C_{2,2,0,2} = 25c_{2,2,0,2}$$
$$C_{2,2,1,1} = -50c_{2,2,0,2}, \quad C_{2,2,2,0} = 5c_{2,2,0,2}$$

Table 5. Coefficients $S_{m,i,j,k}$

$$S_{0,0,0,0} = 1, \quad S_{1,0,0,0} = s_{1,0,0,0}$$
$$S_{1,0,0,1} = \frac{11}{15}S_{1,0,1,0} = \frac{11}{4}, \quad S_{1,1,0,1} = -\frac{1}{3}S_{1,1,1,0} = \frac{3}{4}$$
$$S_{1,1,0,0} = S_{2,1,0,0} = S_{2,2,0,0} = S_{2,2,0,1} = S_{2,2,1,0} = 0$$
$$S_{2,0,0,0} = 3s_{2,0,0,0}$$
$$S_{2,0,0,1} = \frac{-1574EK+193K^2+1480E^2}{1152(E-K)^2}$$
$$S_{2,0,0,2} = \frac{-133892EK+56701K^2+75220E^2}{196608(E-K)^2}$$
$$S_{2,0,1,0} = -\frac{-1226EK+127K^2+1144E^2}{384(E-K)^2}$$
$$S_{2,0,1,1} = -\frac{-51436EK+14687K^2+46172E^2}{98304(E-K)^2}$$
$$S_{2,0,2,0} = -\frac{-16412EK+139K^2+25804E^2}{196608(E-K)^2}$$
$$S_{2,1,0,1} = -\frac{1}{4}n_{2,0,1}$$
$$S_{2,1,0,2} = \frac{3(-1496EK+637K^2+832E^2)}{16384(E-K)^2}$$
$$S_{2,1,1,0} = -\frac{-254EK+13K^2+232E^2}{128(E-K)^2}$$
$$S_{2,1,1,1} = \frac{3(-7172EK+2719K^2+4804E^2)}{16384(E-K)^2}$$
$$S_{2,1,2,0} = -9c_{2,1,0,2}$$
$$S_{2,2,0,2} = \frac{1}{5}S_{2,2,2,0} = -\frac{1}{10}S_{2,2,1,1} = C_{2,2,2,0}$$

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Table 6. Coefficients $\kappa_{m,i,j,k}$

$\kappa_{0,1,0,0} = \frac{3}{4}$
$\kappa_{1,1,0,0} = -\frac{7(64EK-128K^2+64E^2+27)}{192(E-K)^2}$
$\kappa_{1,1,0,1} = \frac{5}{12}n_{1,0,1}, \qquad \kappa_{1,1,1,0} = \frac{17}{12}n_{1,0,1}$
$\kappa_{1,2,0,0} = \kappa_{2,2,0,0} = \kappa_{2,3,0,0} = \kappa_{2,3,0,1} = \kappa_{2,3,1,0} = 0$
$\kappa_{1,2,0,1} = -\kappa_{1,2,1,0} = \frac{13}{24}n_{1,0,1}$
$\kappa_{2,1,0,0} = -\frac{9}{4}n_{2,0,0}$
$\kappa_{2,1,0,1} = -\frac{-3446EK+337K^2+3208E^2}{1152(E-K)^2}$
$\kappa_{2,1,0,2} = \frac{7(-2948EK+1237K^2+1684E^2)}{262144(E-K)^2}$
$\kappa_{2,1,1,0} = -3S_{2,0,0,1}$
$\kappa_{2,1,1,1} = -\frac{-153604EK+52853K^2+115988E^2}{131072(E-K)^2}$
$\kappa_{2,1,2,0} = \frac{9(-4796EK+3067K^2+172E^2)}{262144(E-K)^2}$
$\kappa_{2,2,0,1} = \frac{-6842EK+559K^2+6328E^2}{2304(E-K)^2}$
$\kappa_{2,2,0,2} = -\kappa_{2,2,2,0} = \frac{27(-572EK+251K^2+300E^2)}{32768(E-K)^2}$
$\kappa_{2,2,1,0} = \frac{7}{6}S_{2,1,1,0}, \qquad \kappa_{2,2,1,1} = \frac{416}{25}\kappa^2_{1,1,0,1}$
$\kappa_{2,3,0,2} = \kappa_{2,3,2,0} = \frac{-202268EK+81907K^2+122764E^2}{786432(E-K)^2}$
$\kappa_{2,3,1,1} = -\frac{10}{3}\kappa_{2,3,0,2}$

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Table 7. Coefficients $\sigma_{m,i,j,k}$

$\sigma_{0,1,0,0} = \sigma_{1,1,0,0} = \sigma_{1,2,0,0} = \sigma_{1,2,0,1} = \sigma_{1,2,1,0} = 0$
$\sigma_{2,1,0,0} = \sigma_{2,1,1,0} = \sigma_{2,1,1,1} = \sigma_{2,2,0,0} = \sigma_{2,2,0,1} = 0$
$\sigma_{2,2,1,0} = \sigma_{2,3,0,0} = \sigma_{2,3,0,1} = \sigma_{2,3,0,2} = \sigma_{2,3,1,0} = 0$
$\sigma_{2,3,1,1} = \sigma_{2,3,2,0} = 0, \qquad \sigma_{0,1,0,1} = -\sigma_{0,1,1,0} = \frac{3}{8}$
$\sigma_{1,1,0,1} = -\frac{128E^2+176KE-304K^2+63}{96(E-K)^2}$
$\sigma_{1,1,0,2} = \frac{5}{6}n_{1,0,1}$
$\sigma_{1,1,1,0} = \frac{64E^2+32KE-96K^2+21}{64(E-K)^2}$
$\sigma_{1,1,1,1} = \frac{3}{2}n_{1,0,1}, \qquad \sigma_{1,1,2,0} = -\frac{1}{3}n_{1,0,1}$
$\sigma_{1,2,0,2} = \frac{1}{4}\kappa_{1,2,0,1}, \qquad \sigma_{1,2,1,1} = -6\sigma_{1,2,0,2}$
$\sigma_{1,2,2,0} = \sigma_{1,2,0,2}, \qquad \sigma_{2,1,0,1} = -3n_{2,0,0}$
$\sigma_{2,1,0,2} = \frac{8(4E-K)}{15(E-K)}\kappa_{1,1,0,1}$
$\sigma_{2,1,0,3} = \frac{298364E^2-510796KE+211631K^2}{524288(E-K)^2}$
$\sigma_{2,1,1,2} = \frac{136636E^2-358028KE+178111K^2}{524288(E-K)^2}$
$\sigma_{2,1,2,0} = \frac{4}{3}\sigma_{2,1,0,2}$
$\sigma_{2,1,2,1} = \frac{3(138092E^2-233596KE+96107K^2)}{524288(E-K)^2}$
$\sigma_{2,1,3,0} = \frac{3(15916E^2-17468KE+4891K^2)}{524288(E-K)^2}$
$\sigma_{2,2,0,2} = \frac{392E^2-422KE+39K^2}{512(E-K)^2}$
$\sigma_{2,2,0,3} = \frac{10420E^2-17732KE+7321K^2}{65536(E-K)^2}$
$\sigma_{2,2,1,1} = -\frac{700E^2-761KE+52K^2}{192(E-K)^2}$
$\sigma_{2,2,1,2} = \frac{3(9812E^2-13156KE+4577K^2)}{65536(E-K)^2}$
$\sigma_{2,2,2,0} = \frac{2072E^2-2290KE+65K^2}{4608(E-K)^2}$
$\sigma_{2,2,2,1} = -\frac{3(20764E^2-34892KE+14299K^2)}{65536(E-K)^2}$
$\sigma_{2,2,3,0} = \frac{3(76E^2-572KE+343K^2)}{65536(E-K)^2}$
$\sigma_{2,3,0,3} = -\sigma_{2,3,3,0} = \frac{1}{6}\kappa_{2,3,0,2}$
$\sigma_{2,3,1,2} = -\sigma_{2,3,2,1} = -\frac{5}{2}\kappa_{2,3,0,2}$

Table 8. Coefficients $d_{m,j,k}$

$d_{0,0,0} = -\frac{1-4K^2}{(E-K)^2}$
$d_{0,0,1} = d_{0,1,0} = \frac{3}{4}, \qquad d_{1,0,0} = d_{2,0,0} = d_{2,1,0} = 0$
$d_{1,0,1} = -\frac{16EK-272K^2+256E^2+63}{48(E-K)^2}$
$d_{1,0,2} = \frac{9}{8}n_{1,0,1}$
$d_{1,1,0} = -\frac{7(-32EK-32K^2+64E^2+9)}{96(E-K)^2}$
$d_{1,1,1} = 2d_{1,2,0} = \frac{1}{4}n_{1,0,1}, \qquad d_{2,0,1} = 6n_{2,0,0}$
$d_{2,0,2} = -\frac{-1730EK+205K^2+1624E^2}{384(K-E)^2}$
$d_{2,0,3} = \frac{-55220EK+22837K^2+32356E^2}{49152(K-E)^2}$
$d_{2,1,1} = -\frac{-799EK+68K^2+740E^2}{48(K-E)^2}$
$d_{2,1,2} = \frac{-1364EK+733K^2+388E^2}{16384(K-E)^2}$
$d_{2,2,0} = -\frac{-1382EK+139K^2+1288E^2}{384(K-E)^2}$
$d_{2,2,1} = 3d_{2,3,0} = -\frac{-2332EK+719K^2+1964E^2}{16384(K-E)^2}$