heyoka: Modern Numerical Integration and its uses in Astrodynamics and Spaceflight Mechanics

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September 14, 2022

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What is heyoka anyway?

heyoka is a software package for the numerical integration of ODEs (ordinary differential equations) via Taylor's method

https://github.com/bluescarni/heyoka
https://github.com/bluescarni/heyoka.py

Heyoka

Use in ACT projects

- G&CNETs + Backward Generation of Optimal Examples
- Poincaré maps classification
- Dark matter inversion
- EclipseNET
- Kelvins space debris competition

Why should I care?! I can use ...

Main motivations

Only one actively-maintained implementation of Taylor's method

Superior performance (speed & accuracy)

Unique features enabling novel applications

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Free dense output with guaranteed precision

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Symbolic differentiation: cumbersome, error-prone, exponential complexity

Automatic differentiation (AD) to the rescue:

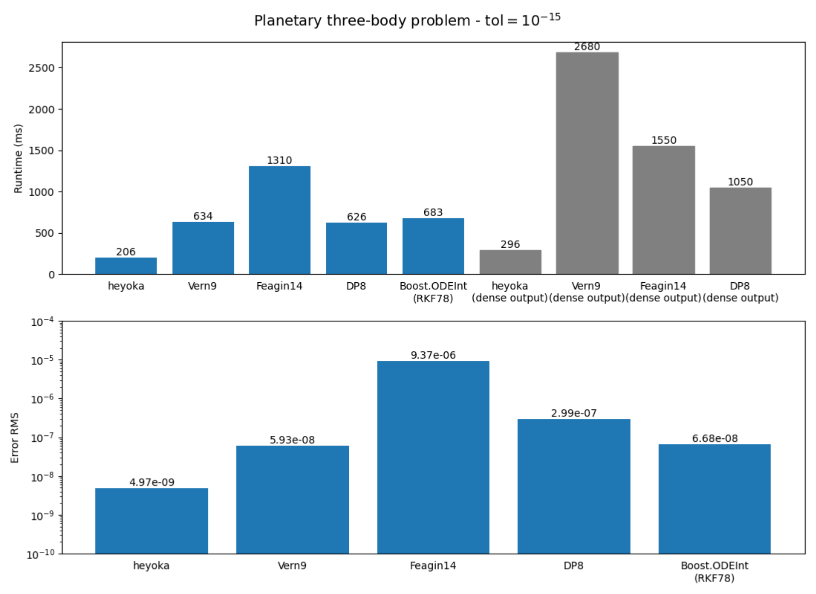
1. decompose $f(x(t))$ into a graph of elementary subexpressions
2. apply AD rules on the subexpressions

However, the implementation is technically very challenging

heyoka – The sales pitch

- Novel implementation of Taylor's method based on a just-in-time (JIT) compilation approach
- Batch mode to fully utilise modern vector instruction sets
- Support for coarse-grained and fine-grained automatic parallelisation
- Support for extended-precision arithmetic
- Optimally accurate
- Support for reliable and accurate event-detection

Obligatory benchmark slide



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Existing approaches leave much to be desired ...

There's gotta be a better way...

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Taylor series of the event function (via AD):

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Use dense output to propagate the state of the system up to the trigger time

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Preliminary results:

- Guaranteed collision detection for cm-sized objects moving at km/s speeds
- No constraints on the integration timestep
- $\sim$ 4 hours for $\sim$ 20k objects, 20 years (64 cores)

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Still a work-in-progress...

Questions?