COMPTON SCATTERING

The Klein Nishina cross section:

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} r_0^2 \frac{h\nu^2}{h\nu_0^2} \left[\frac{h\nu_0}{h\nu} + \frac{h\nu}{h\nu_0} - 2 + 4\cos^2\Theta \right]$$

Where,

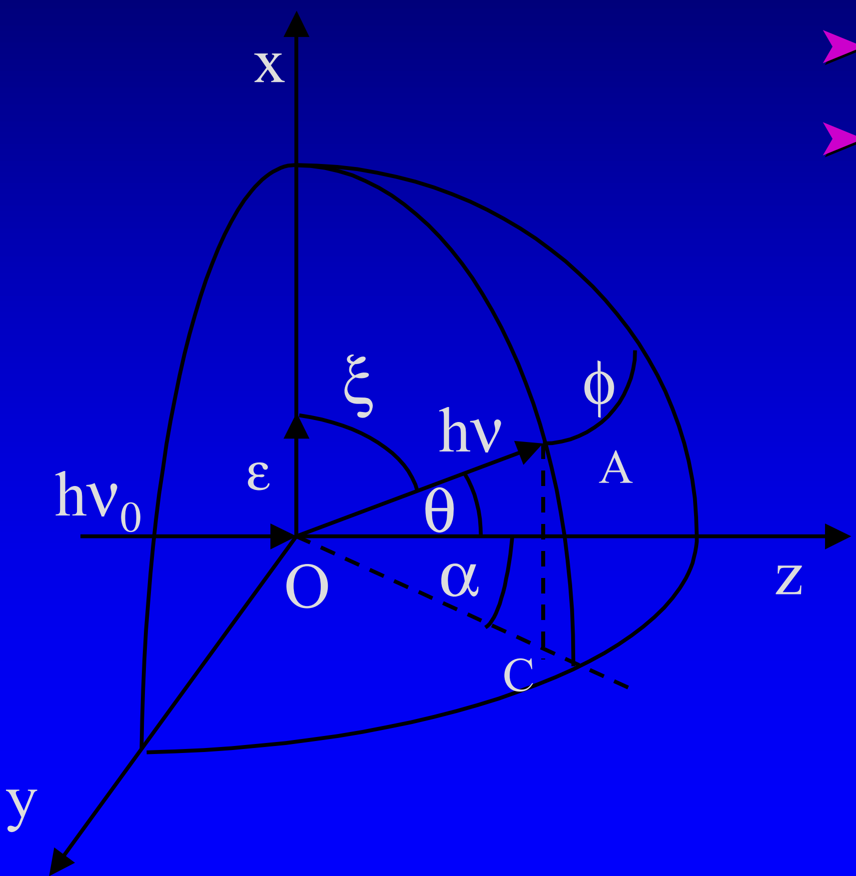
$h\nu_0$: energy of incident photon.

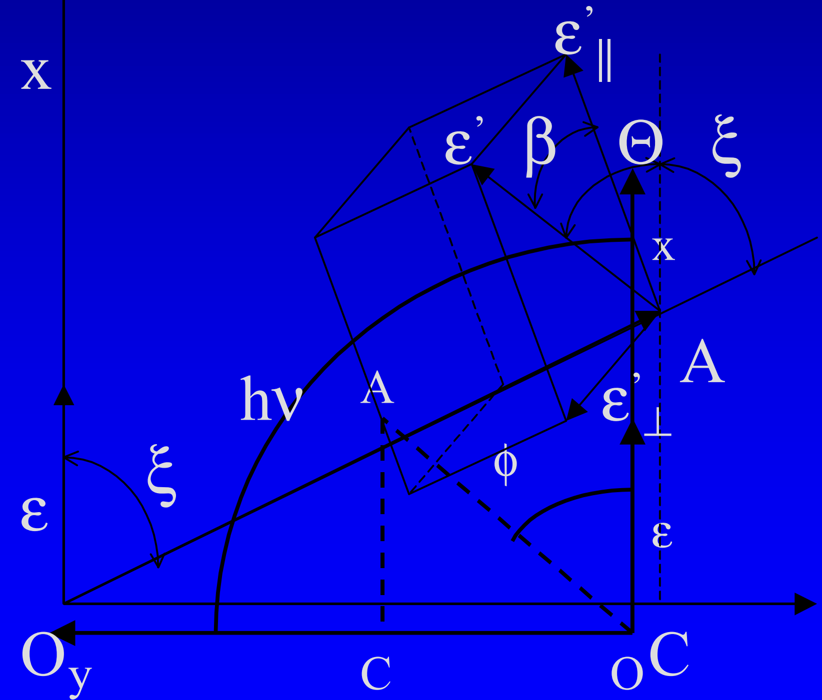
$h\nu$: energy of the scattered photon.

Θ : angle between the two polarization vectors

Angles in the Compton Effect

- θ Polar angle
- ϕ Azimuthal angle
- ε Polarization vector





Summing over the two directions $\longrightarrow$ the cross section can be written as:

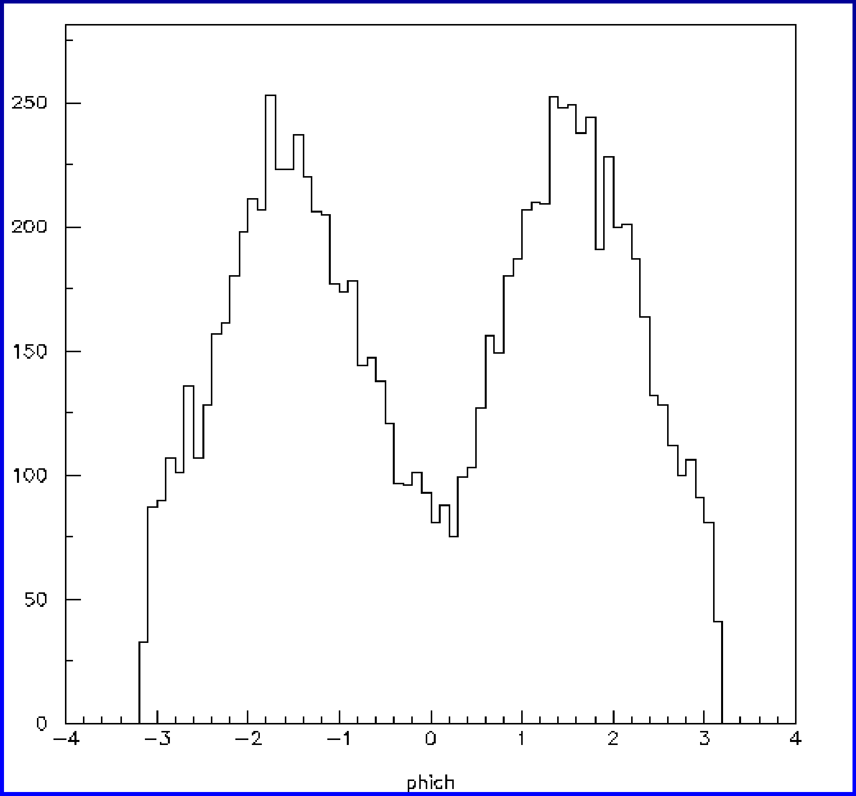
$$\frac{d\sigma}{d\Omega} = \frac{1}{2} r_0^2 \frac{h\nu^2}{h\nu_0^2} \left[\frac{h\nu_0}{h\nu} + \frac{h\nu}{h\nu_0} - 2\sin^2\theta\cos^2\phi \right]$$

Sample Methods implemented in G4LowEnergyPolarizedCompton class :

- Integrate over ϕ
- Sample θ
- Theta - Energy Relation $\longrightarrow$ Energy
- Sample of ϕ from $P(\phi) = a\,(b - c\cos^2\phi)$ distribution

Results

ϕ distribution obtained with the class



Scattered Photon Polarization

$$\cos\xi = \sin\theta\cos\phi \quad \Rightarrow \quad \sin\xi = \sqrt{1-\sin^2\theta\cos^2\phi} = N$$

$$\overline{\varepsilon'_\perp} = \frac{1}{N}\left(\cos\theta\,\hat{j} - \sin\theta\sin\phi\,\hat{k}\right)\sin\beta$$

$$\overline{\varepsilon'_\parallel} = \left(N\hat{i} - \frac{1}{N}\sin^2\theta\,\sin\phi\,\cos\phi\,\hat{j} - \frac{1}{N}\sin\theta\,\cos\theta\,\cos\phi\,\hat{k}\right)\cos\beta$$

β is obtained from $\cos\Theta = \cos\beta\, N$ and Θ is sampled from Klein Nishina cross section

The Stokes Parameters

Introduced by G. G. Stokes in 1852 to characterise a beam of light. They are represented by a 4-vector:

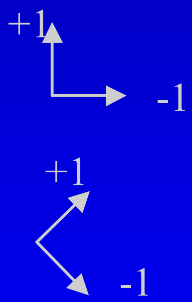
$$\begin{bmatrix} I \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} I \\ \mathbf{P} \end{bmatrix}$$

where: I is the intensity; P_1 is the degree of plane polarization with respect to arbitrary orthogonal axes; P_2 is the degree of plane polarization with respect to a set of axes oriented at $\pi/4$ to right of the previous one and P_3 is the degree of circular polarization.

Examples of characterization of a beam by the Stokes parameters:

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ represent an unpolarized beam;

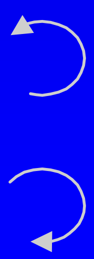
$\begin{bmatrix} 1 \\ \pm 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ \pm 1 \\ 0 \end{bmatrix}$ represented plane polarization.



Plane polarization

Plane polarization at $\pi/4$ to The right

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \pm 1 \end{bmatrix}$ represent circular polarization



+1 left circular polarization

-1 right circular polarization

Interactions with the Stokes Parameters:

When photons undergo an interaction, in general the Stokes parameters of the initial beam are transformed into a new set of parameters. The relation between these two Stokes vectors is given by a transformation matrix T characteristic of the interaction. That is:

$$\begin{bmatrix} I \\ \mathbf{P} \end{bmatrix} = T \begin{bmatrix} I_0 \\ \mathbf{P_0} \end{bmatrix}$$

In this way, the fractional intensity detected with an analyzer characterized by the parameters (1,**D**) of a beam characterized by (I, **P**) that it transformed by an interaction represented by matrix T is:

$$W = \frac{1}{2} \begin{bmatrix} 1 & \mathbf{D} \end{bmatrix} T \begin{bmatrix} I \\ \mathbf{P} \end{bmatrix}$$

Matrix transformation for the Compton scattering

$$T = \frac{1}{2} r_0^2 \left(\frac{k}{k_0} \right)^2 \begin{bmatrix} 1+\cos^2\theta + (k_0 - k)(1-\cos\theta) & \sin^2\theta & 0 & -(1-\cos\theta)(\mathbf{k_0}\cos\theta + \mathbf{k})\bullet\mathbf{S} \\ \sin^2\theta & 1+\cos^2\theta & 0 & (1-\cos\theta)(\mathbf{n}\times\mathbf{n_0})\bullet(\mathbf{k_0}\times\mathbf{S}) \\ 0 & 0 & 2\cos\theta & (1-\cos\theta)(\mathbf{k_0}\times\mathbf{n})\bullet\mathbf{S} \\ -(1-\cos\theta)(\mathbf{k}\cos\theta + \mathbf{k_0})\bullet\mathbf{S} & (1-\cos\theta)(\mathbf{n_0}\times\mathbf{n})\bullet(\mathbf{k}\times\mathbf{S}) & (1-\cos\theta)(\mathbf{k}\times\mathbf{n_0})\bullet\mathbf{S} & 2\cos\theta + (\mathbf{k_0} - \mathbf{k})(1-\cos\theta)\cos\theta \end{bmatrix}$$

Use of the Stokes Parameters. Example 1: Compton Scattering of Unpolarized Gamma rays.

The Stokes parameters of an unpolarized bean undergo after Compton Scattering into:

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} \sim \begin{bmatrix}1+\cos^2\theta+(k_0-k)(1-\cos\theta)\\ \sin^2\theta\\ 0\end{bmatrix}$$

From the definition of the degree of polarization:

$$P=\frac{\sqrt{P_1^2+P_2^2+P_3^2}}{I}$$

One can see the scattered radiation of an unpolarized gamma ray will have some degree of polarization:

$$P=\frac{sin^2\,\theta}{1+cos^2\,\theta+(k_0-k)(1-cos\,\theta)}$$

Sometimes is more easy, due the experimental setup, to measure the ratio of intensities with opposite polarization in the scattered beam, this ratio is defined by:

$$p = \frac{d\sigma_{\perp} / d\Omega}{d\sigma_{||} / d\Omega}$$

where:

$$\frac{d\sigma_{\perp}}{d\Omega} = \frac{1}{2}\begin{bmatrix}1 & 1 & 0\end{bmatrix} T \begin{bmatrix}1 \\ 0 \\ 0\end{bmatrix} = \frac{1}{4} r_0^2 \left(\frac{k}{k_0}\right)^2 \left[(k_0 - k)(1 - cos\,\theta) + 2\right] = \frac{1}{4} r_0^2 \left(\frac{k}{k_0}\right)^2 \left[\frac{k}{k_0} + \frac{k_0}{k}\right]$$

$$\frac{d\sigma_{||}}{d\Omega} = \frac{1}{2}\begin{bmatrix}1 & -1 & 0\end{bmatrix} T \begin{bmatrix}1 \\ 0 \\ 0\end{bmatrix} = \frac{1}{4} r_0^2 \left(\frac{k}{k_0}\right)^2 \left[(k_0 - k)(1 - cos\,\theta) + 2\,cos^2\,\theta\right] = \frac{1}{4} r_0^2 \left(\frac{k}{k_0}\right)^2 \left[\frac{k}{k_0} + \frac{k_0}{k} - 2\,sin^2\,\theta\right]$$

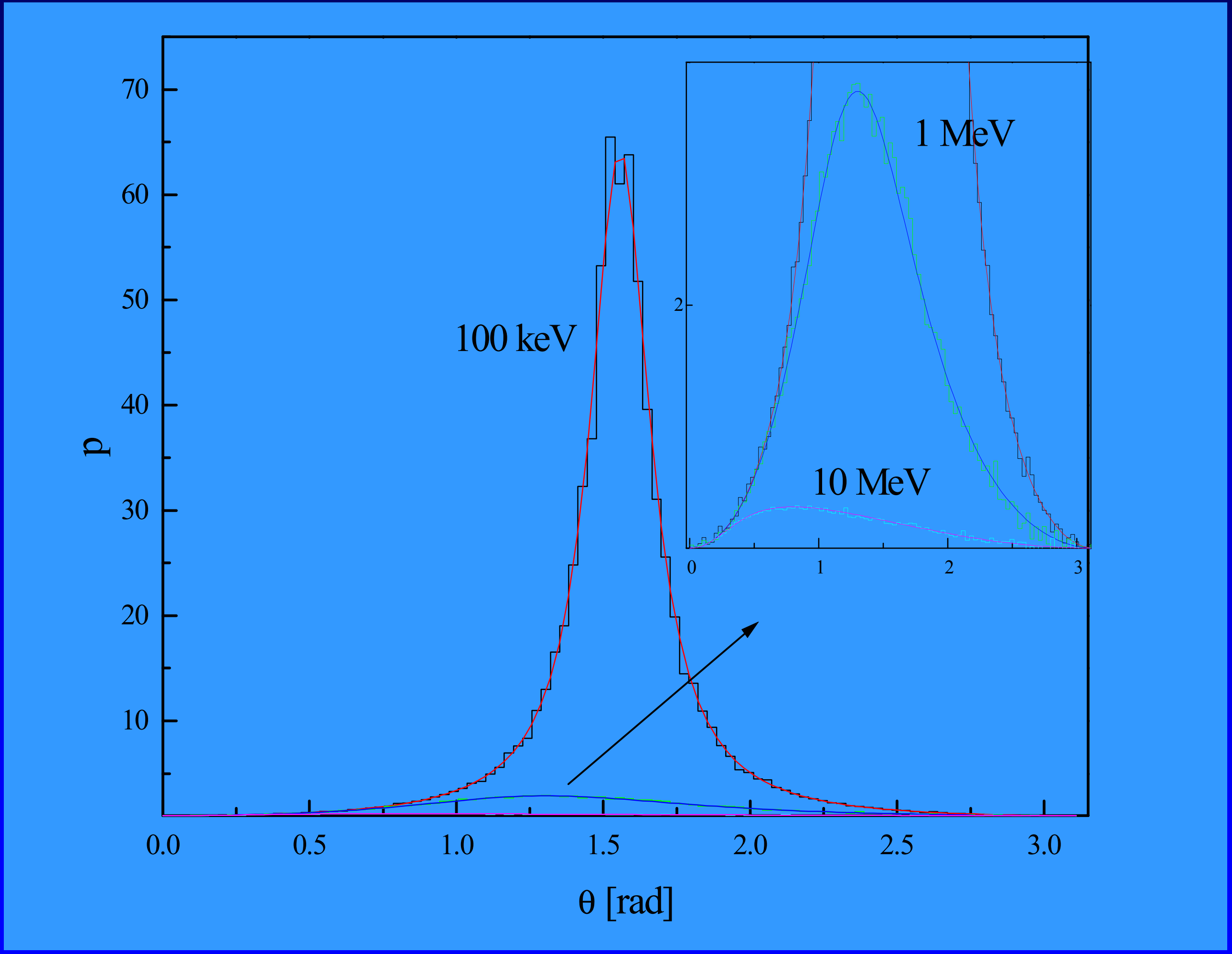
where the || and ⊥ refer to plane polarization parallel to and perpendicular to the plane of scattering, respectively.

yielding the result:

$$p = \frac{(k_0 - k)(1 - cos\,\theta) + 2}{(k_0 - k)(1 - cos\,\theta) + 2\,cos^2\,\theta}$$

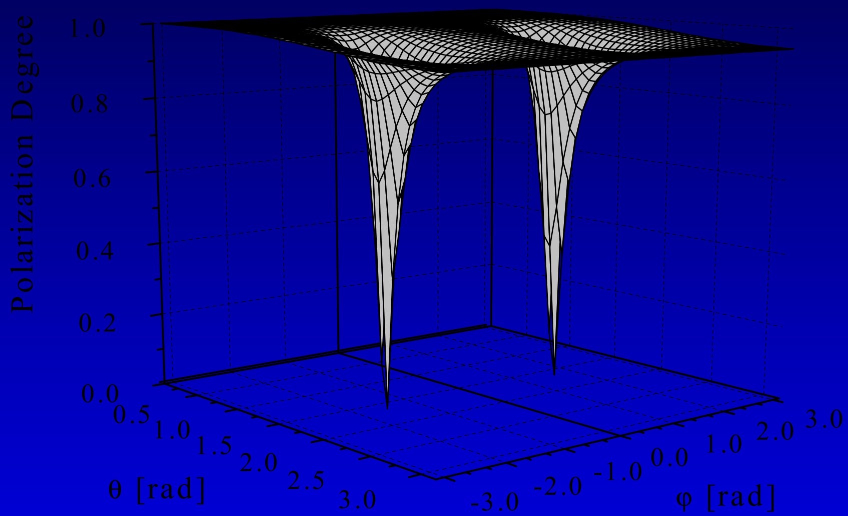
For $\theta = 0$ and π we have $p = 1$, that is, an unpolarized beam.

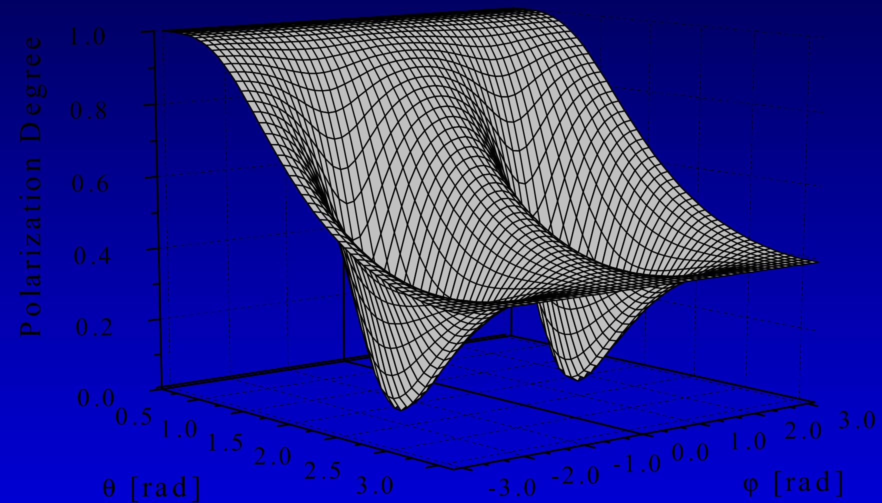
Comparison between the theoretical rates of intensities with that obtained from GEANT4, for 3 energies: 100 keV, 1 MeV and 10 MeV.

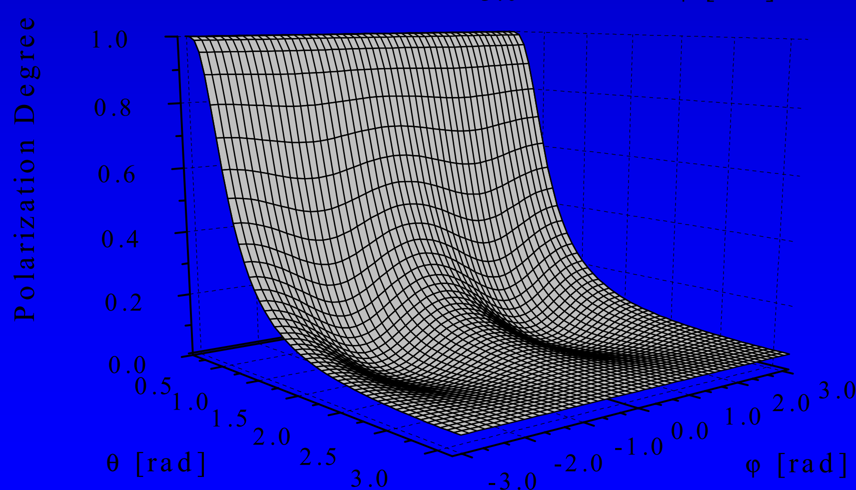


Example 2: Compton Scattering of Polarized Gamma rays.

$$P = 2\frac{1 - sin^2\theta\, sin^2\varphi}{(k_0 - k)(1 - cos\theta) + 2 - 2\, sin^2\theta\, sin^2\varphi}$$





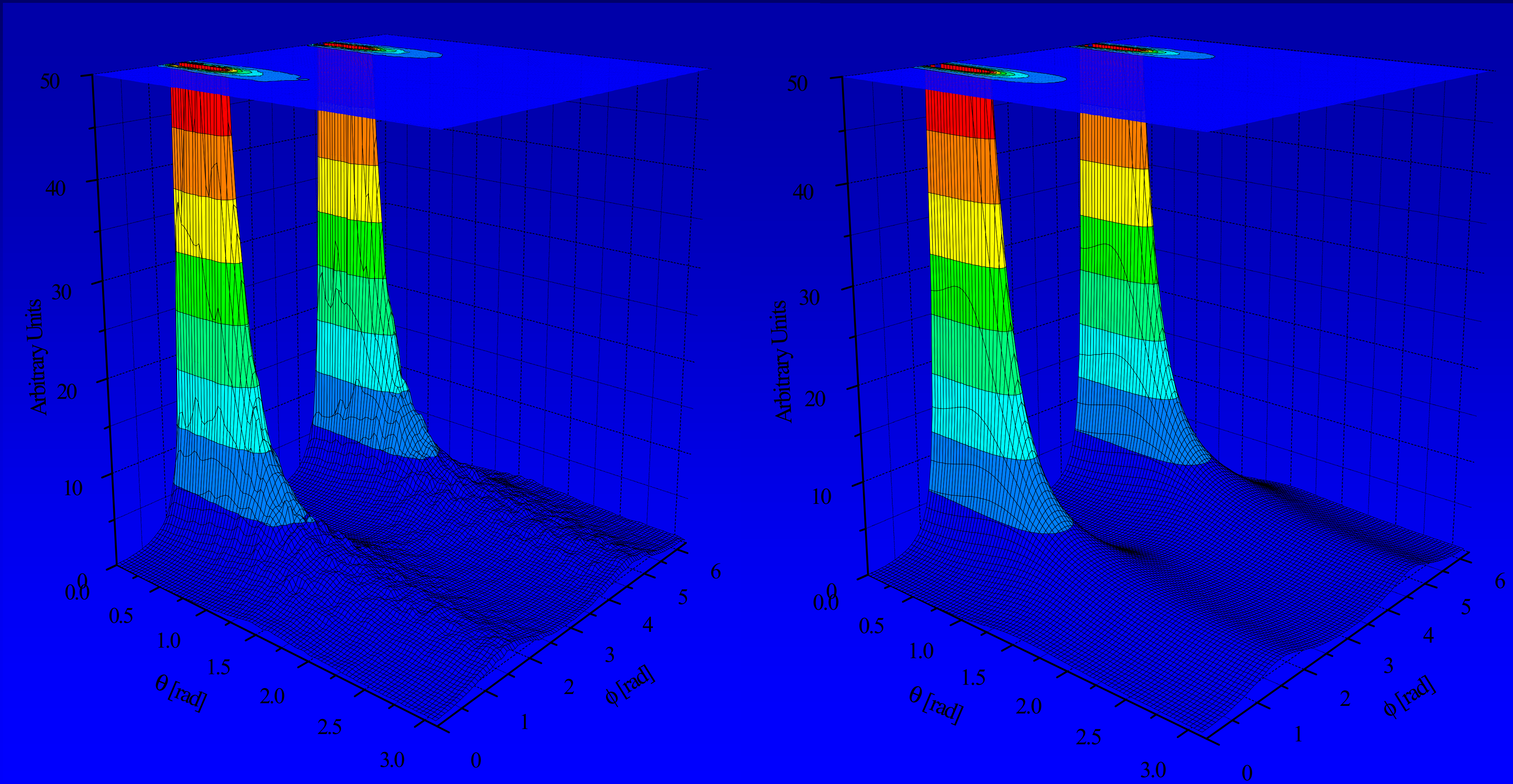


Degree of polarization for scattered radiation by incident linearly polarized gamma ray of 100 keV (top left), 1 MeV (top right) and 10 MeV (left) as function of θ and φ.

Comparison between the theoretical rates of intensities with that obtained from GEANT4 for 1 MeV Gamma rays.

Simulate

Theoretical: $$p = \frac{(k_0 - k)(1 - \cos\theta) + 4\cos^2\varphi}{(k_0 - k)(1 - \cos\theta) + 4\cos^2\theta\sin^2\varphi}$$



Binding effects

The above relations are valid for scattering by free electrons at rest. For unpolarized photon, the atomic binding effects are usually taken into account by introducing the incoherent scattering function S, with depends on the momentum transfer $x = (k_0/hc)\sin(\theta/2)$ and the atomic number Z. As a theory describing the polarization-dependent binding effects is lacking at present, we assume, as it is usually done, that S(x,Z) is polarization–independent, an therefore:

$$\frac{d\sigma}{d\Omega} = S(x,z)\frac{d\sigma_{KN}}{d\Omega}$$

(here $d\sigma_{KN}/d\Omega$ is the Klein-Nishina cross section)

In practice, this means that the scattering angle θ must be determinated according to the form factor; ϕ, instead, is assumed to be unaffected by the binding effects, and can be calculated in the same way as for free electrons.

Example of application: MEGA instrument

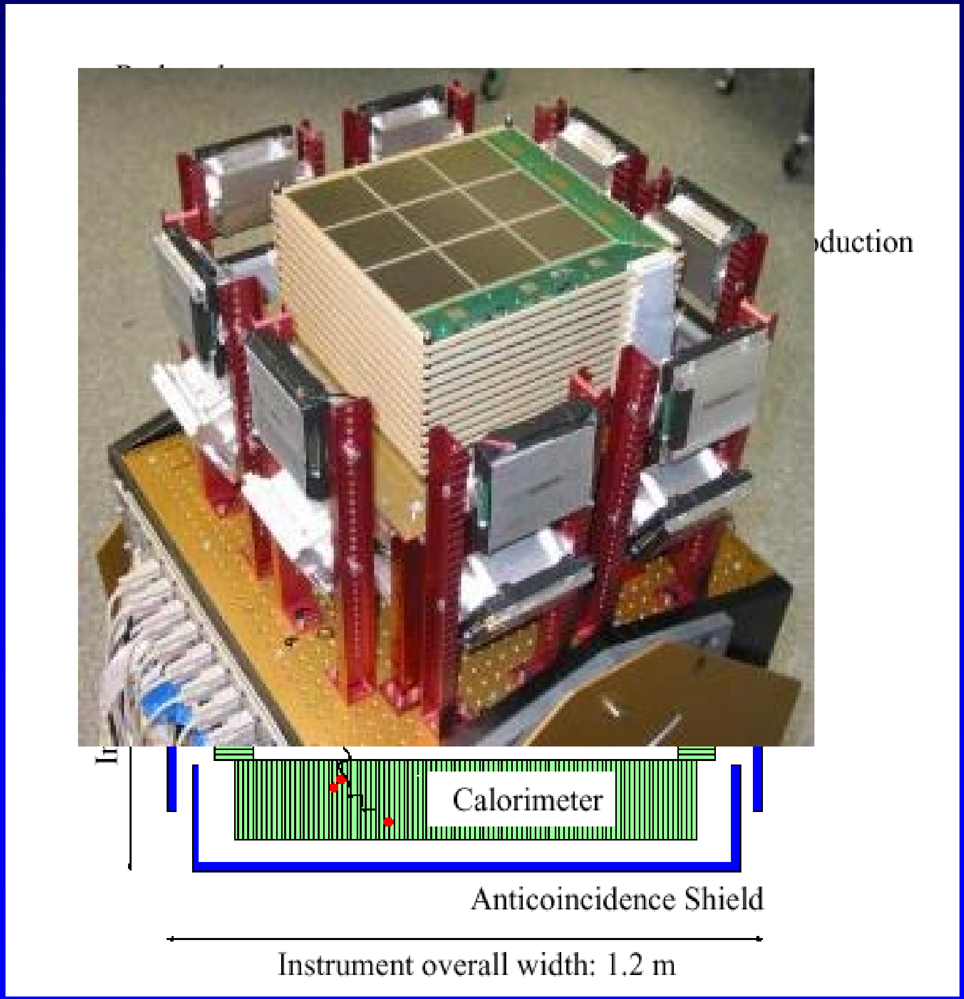
Compton Telescope developed at MPE - Münichen - Germany

Tracker: Planes of double side silicon strip detector.

Thick = 0.5 mm
Pitch = 470 µm.

Calorimeter: CsI crystal bars
Bottom = 5.5 x 5.5 x 80 mm^3.
Lateral = 5.5 x 5.5 x 40 mm^3.

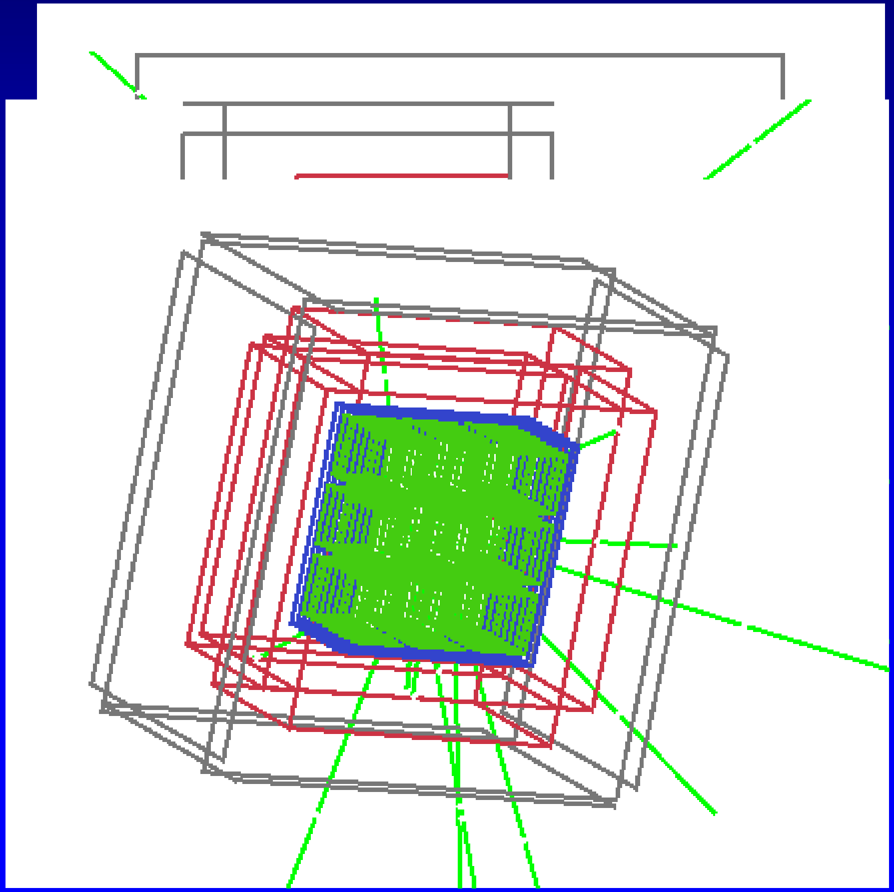
Anticoincidence Shield: plastic scintillator with a wall thickness of 15 mm



Simulation:

Photon energy: 100 keV

Polarized photons



Unpolarized photons